c. 2

RESEARCH MEMORANDUM

DERIVATION OF THE EQUATIONS OF MOTION OF A SYMMETRICAL WING-TIP-COUPLED AIRPLANE CONFIGURATION WITH ROTATIONAL FREEDOM AT THE JUNCTURES

By Albert A. Schy

Langley Aeronautical Laboratory
Langley Field, Va.

NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

WASHINGTON
October 8, 1951

NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

RESEARCH MEMORANDUM

DERIVATION OF THE EQUATIONS OF MOTION OF A SYMMETRICAL WING-TIP-COUPLED AIRPLANE CONFIGURATION WITH ROTATIONAL FREEDOM AT THE JUNCTURES

By Albert A. Schy

SUMMARY

The method of Lagrange multipliers is used to take account of the dynamic effects of the constraints at the wing tips when two identical airplanes are coupled to the wing tips of a "mother" airplane. The resulting equations of motion of this symmetrical configuration are derived for one, two, or three degrees of rotational freedom at each joint.

The effects of aerodynamic interference are ignored in the original derivation. It is then shown how the aerodynamic interference terms may be included when they are known. As an example, the interference terms arising from variations of the rolling velocities and angles of attack of the individual airplanes are treated.

INTRODUCTION

Wing-tip to wing-tip coupling of airplanes is being investigated as a means for carrying fighter protection on bomber missions or for in-flight refueling. Because of the increase in effective aspect ratio in this configuration, the outer "parasite" airplanes can probably be carried more efficiently than in any other manner (references 1 to 4). The method of tandem coupling does not have this advantage of increased aspect ratio and has, in addition, proven to be inefficient because the rear airplane must fly in the downwash of the front airplane.

The structural loads on the wings in the wing-tip-coupled configuration may be minimized by allowing rotational freedom at the joints. In this paper it is shown how the method of Lagrange multipliers may be used to analyze the dynamic effects of the constraints at the wing tips.

[redacted]

The resulting equations of motion for small disturbances of a symmetrical configuration from its trim condition are derived for one, two, or three degrees of rotational freedom at each joint. Because of the symmetry of the configuration, the equations may be separated into independent lateral and longitudinal modes.

The primary purpose of this investigation is to analyze the purely dynamic effects of wing-tip coupling; therefore the equations of motion are first derived without considering the aerodynamic interference between the airplanes. Such interference effects, however, may be important. Therefore, the types of aerodynamic interference terms which may occur are discussed from a general point of view, and it is shown how these terms, when they are known, can be included in the equations of motion. As an example, it is shown how the interference terms arising from variations of the rolling velocities and angles of attack of the individual airplanes may be included in the equations of motion. These terms are believed to be the most important interference effects in this type of coupling.

The effects of aeroelasticity are ignored in this discussion.

SYMBOLS

X, Y, Z	conventional stability axes for describing airplane motions; components of aerodynamic forces along these axes; also components of fixed positions of coupling joints in these axes
V_o	steady-state velocity, taken along the steady-state X-axis
t	time
x, y, z	components of disturbance displacement of airplane along X-, Y-, and Z-axes, respectively
u, v, w	components of disturbance translational velocity along X-, Y-, and Z-axes, respectively (dx/dt, dy/dt, dz/dt taken in stability axes)
u', β, α	nondimensional forward velocity, sideslip angle, and angle of attack, respectively
ϕ, θ, ψ	components of disturbance rotation about X-, Y-, and Z-axes, respectively

λ_k	an arbitrary factor (Lagrange's "undetermined multipliers")
δ	indicates a small virtual increment
i	index denoting degrees of freedom of a mechanical system
q_i	generalized coordinates used to describe a mechanical system; also used, in particular, to denote degrees of freedom of the three airplanes
F_i	generalized applied force in airplane degree of freedom
E_i	sum of the inertial, weight, and aerodynamic forces in unconstrained equations of motion of an airplane
m	mass
W	weight
I_X, I_Y, I_Z	moments of inertia of airplane about the X-, Y-, and Z-axes, respectively
I_{XZ}	product of inertia of airplane
γ	flight-path angle
L'	lift force
b	wing span
c	wing chord
S	wing area
ρ	air density
$q = \frac{1}{2}\rho V_o^2$	
F_X, F_Y, F_Z	components of applied forces along stability axes
L, M, N	components of applied moments about X-, Y-, and Z-axes, respectively; also components of aerodynamic moments
σ	angle of fighter deflection about hinge, when hinge-type coupling is used

η_o angle between hinge axis and steady-state X-axis when hinge axis is parallel to steady-state XZ-plane

η_1, η_2, η_3 angles between hinge axis and the steady-state X-, Y-, and Z-axes, respectively, when the hinge is arbitrarily oriented

$$A_1 = \frac{S_f b_f}{Sb}$$

$$A_2 = \frac{S_f {b_f}^3}{Sb^3}$$

$$A_3 = \frac{S_f}{S}$$

$$A_4 = \frac{b_f}{b}$$

$$B_1 = \frac{S_f c_f}{Sc}$$

$$B_2 = \frac{S_f {c_f}^3}{Sc^3}$$

$$B_3 = \frac{S_f}{S}$$

$$B_4 = \frac{c_f}{c}$$

$$B_5 = \frac{b_f}{c}$$

$$B_6 = \frac{b_f}{c_f}$$

$$D_b = \frac{b}{V_o} \frac{d}{dt}$$

$$D_c = \frac{c}{V_o} \frac{d}{dt}$$

$K_X{}^2$, $K_Y{}^2$, $K_Z{}^2$ square of the nondimensional radii of gyration in roll, pitch, and yaw, respectively $\left(\frac{I_X}{mb^2}, \frac{I_Y}{mc^2}, \frac{I_Z}{mb^2}\right)$

K_{XZ} nondimensional product-of-inertia parameter $\left(\frac{-I_{XZ}}{mb^2}\right)$

$\mu_b = \frac{m}{\rho Sb}$

$\mu_c = \frac{m}{\rho Sc}$

$C_W = \frac{W}{qS} \cos \gamma_o$

C_X longitudinal-force coefficient $\left(\frac{X}{qS}\right)$

C_Y lateral-force coefficient $\left(\frac{Y}{qS}\right)$

C_Z normal-force coefficient $\left(\frac{Z}{qS}\right)$

C_l rolling-moment coefficient $\left(\frac{L}{qSb}\right)$

C_m pitching-moment coefficient $\left(\frac{M}{qSc}\right)$

C_n yawing-moment coefficient $\left(\frac{N}{qSb}\right)$

$$C_{Z_q} = \frac{\partial C_Z}{\partial\left(\frac{c\dot{\theta}}{2V_o}\right)}$$

$$C_{l_p} = \frac{\partial C_l}{\partial\left(\frac{b\dot{\phi}}{2V_o}\right)}$$

$$C_{l_r} = \frac{\partial C_l}{\partial\left(\frac{b\dot{\psi}}{2V_o}\right)}$$

$$C_{m_{\dot{\alpha}}} = \frac{\partial C_m}{\partial\left(\frac{c\dot{\alpha}}{2V_o}\right)}$$

$$C_{m_q} = \frac{\partial C_m}{\partial\left(\frac{c\dot{\theta}}{2V_o}\right)}$$

$$C_{n_p} = \frac{\partial C_n}{\partial\left(\frac{b\dot{\phi}}{2V_o}\right)}$$

$$C_{n_r} = \frac{\partial C_n}{\partial\left(\frac{b\dot{\psi}}{2V_o}\right)}$$

Whenever u, u', v, β, w, $\dot{w}$, α, $\dot{\phi}$, $\dot{\theta}$, and $\dot{\psi}$ are used as subscripts, a derivative is indicated. For example, $Y_v = \frac{\partial Y}{\partial v}$ and $C_{Y_\beta} = \frac{\partial C_Y}{\partial \beta}$.

Note: Unconventional stability derivatives caused by aerodynamic interference are distinguished by parentheses; namely,

$(Z)_{\alpha_f}$, $(Z)_{\dot{\phi}_f}$ aerodynamic Z-force on bomber due to unit fighter angle of attack and unit fighter rolling velocity, respectively

$(L)_{\alpha_f}$, $(L)_{\dot{\phi}_f}$ aerodynamic rolling moment on bomber due to unit fighter angle of attack and unit fighter rolling velocity, respectively

$(Z_f)_{\alpha}$, $(Z_f)_{\dot{\phi}}$ aerodynamic Z-force on fighter due to unit bomber angle of attack and unit bomber rolling velocity, respectively

$(L_f)_{\alpha}$, $(L_f)_{\dot{\phi}}$ aerodynamic rolling moment on the fighter due to unit bomber angle of attack and unit bomber rolling velocity, respectively

$(Z_{\dot{\phi}})_f$ aerodynamic Z-force on fighter due to unit rolling velocity of same fighter

$(L_\alpha)_f$ aerodynamic rolling moment on fighter due to unit angle of attack of same fighter

Subscripts:

o	initial conditions
f	fighter parameters
1, 2	parameters of left and right fighters, respectively
i	index denoting the degrees of freedom of a mechanical system
k	index denoting a particular equation of condition connecting the degrees of freedom, or the corresponding undetermined multiplier
s, a	symmetrical and antisymmetrical components of combined fighter motion, respectively (see equations (9), (10), and (16))

Note: No subscript is used on bomber parameters.

PRELIMINARY DISCUSSION

The configuration to be considered consists of three airplanes coupled wing tip to wing tip. The two outer airplanes are assumed to be exactly alike. Since the central airplane is carrying the two outer airplanes along as parasites, these airplanes will probably be smaller than the central airplane. For convenience, the central airplane will be referred to as a bomber and the outer airplanes will be referred to as fighters, although the analysis will actually be quite general and the only real restriction will be that the outer airplanes, including their coupling to the central airplane, be exactly alike.

The ordinary equations of motion for the three airplanes flying independently will be modified by the interactions between the airplanes. These interactions are of three types: the dynamic interactions arising from the coupling at the wing tips, the aerodynamic interactions arising from the interference effects on the air flow over the airplanes due to their proximity, and the elastic interactions. In the present case the airplanes are assumed to be rigid; therefore, elastic effects may be ignored.

The primary purpose of the present paper is to take account of the dynamic effects of the constraints at the wing tips and set up a method for obtaining the equations of motion of the entire configuration for small disturbances from a trim condition. This method will first be developed without considering the aerodynamic interference effects. The equations so derived would be correct for airplanes coupled together with a large enough gap between the wing tips so that aerodynamic interference could be ignored. Later, it will be shown how the effects of aerodynamic interference may be considered without any essential modification of the method of obtaining the equations of motion of the configuration.

DERIVATION OF EQUATIONS OF MOTION

WITHOUT AERODYNAMIC INTERFERENCE

In order to describe the motion of three independent airplanes, eighteen degrees of freedom consisting of the ordinary six degrees of freedom for each airplane must be considered. If the wing tips are assumed to be connected, then the translational motion of adjacent wing tips must be the same. The translational motion of the wing tips at each point of connection is expressed in terms of the degrees of freedom of the individual airplanes. Then the expressions for the translational velocities of adjacent wing tips may be set equal to each other. Thus, for each connection, three equations are obtained relating the degrees of freedom of the airplanes, since one equation is obtained for each component of the translational velocity of the wing tips. There are, therefore, six equations of condition (or constraint) connecting the eighteen degrees of freedom for the case of complete rotational freedom at each connection. The equations of motion of the total system, taking account of the effects of the constraints, will be derived using Lagrange's method of undetermined multipliers (references 5 and 6).

The equations of motion for each independent airplane are referred to stability axes fixed in the separate airplanes (see fig. 1). For the present discussion it will be assumed that in the equilibrium condition of the configuration the fighters may be trimmed so that all six wing tips lie on a straight line; then in the steady state all three sets of airplane axes may be taken parallel. For each airplane, the components of the wing-tip velocities along axes fixed in space parallel to the steady-state axes may now be written in terms of airplane degrees of freedom as follows:

$$V_o + u + Z\dot{\theta} - Y\dot{\psi} \tag{1a}$$

$$v + V_o\psi + X\dot{\psi} - Z\dot{\phi} \tag{1b}$$

$$w - V_o\theta + Y\dot{\phi} - X\dot{\theta} \tag{1c}$$

where u, v, w and $\dot{\phi}$, $\dot{\theta}$, $\dot{\psi}$ are the disturbance translational and rotational velocities of the airplane axes, V_o is the steady-state velocity along the X-axis, and X, Y, and Z are the fixed positions of the wing tip in airplane coordinates.

The subscript f will be used to indicate the fighter parameters common to both fighters. The fighter to the left of the bomber will be indicated by using the subscript 1 on its variables and the fighter to the right of the bomber will be indicated by using the subscript 2. No subscript will be used for the bomber. The constraining conditions are that the velocity of the right wing tip of the left fighter must equal the velocity of the left wing tip of the bomber and the velocity of the left wing tip of the right fighter must equal the velocity of the right wing tip of the bomber. For example, the two equations giving the X-velocities of the points of connection are, from expression (1a),

$$V_o + u + Z\dot{\theta} + \frac{b}{2}\dot{\psi} = V_o + u_1 + Z_f\dot{\theta}_1 - \frac{b_f}{2}\dot{\psi}_1$$

$$V_o + u + Z\dot{\theta} - \frac{b}{2}\dot{\psi} = V_o + u_2 + Z_f\dot{\theta}_2 + \frac{b_f}{2}\dot{\psi}_2$$

Solving these equations and the similar equations of condition for the translational velocities of the fighters gives

$$u_1 = u + Z\dot{\theta} + \frac{b}{2}\dot{\psi} - Z_f\dot{\theta}_1 + \frac{b_f}{2}\dot{\psi}_1 \tag{2.1}$$

$$v_1 = v + V_o\psi + X\dot{\psi} - Z\dot{\phi} - V_o\psi_1 - X_f\dot{\psi}_1 + Z_f\dot{\phi}_1 \tag{2.2}$$

$$w_1 = w - V_o\theta - \frac{b}{2}\dot{\phi} - X\dot{\theta} + V_o\theta_1 - \frac{b_f}{2}\dot{\phi}_1 + X_f\dot{\theta}_1 \tag{2.3}$$

$$u_2 = u + Z\dot{\theta} - \frac{b}{2}\dot{\psi} - Z_f\dot{\theta}_2 - \frac{b_f}{2}\dot{\psi}_2 \tag{2.4}$$

$$v_2 = v + V_o\psi + X\dot{\psi} - Z\dot{\phi} - V_o\psi_2 - X_f\dot{\psi}_2 + Z_f\dot{\phi}_2 \tag{2.5}$$

$$w_2 = w - V_o\theta + \frac{b}{2}\dot{\phi} - X\dot{\theta} + V_o\theta_2 + \frac{b_f}{2}\dot{\phi}_2 + X_f\dot{\theta}_2 \tag{2.6}$$

These equations are the equations of condition imposed by the constraints when the axes of all the airplanes are parallel. Inasmuch as small angles occur between the sets of axes during the motion, equations (2) should be modified to be exact by introducing the proper trigonometric functions of these variable angles. However, it is apparent that equations (2) remain valid to first-order accuracy as long as the disturbances are small. Therefore, for the purposes of stability analysis these equations are valid equations of condition if the airplanes have been alined as previously described in the trim condition. Actually, other possible trim conditions exist in which there may be steady-state angles between the bomber and fighter; therefore, the bomber and fighter axes are inclined at an angle to each other. In such cases the proper trigonometric functions of the steady-state angles must be introduced into the equations of condition. Since the presence of these constant factors does not fundamentally alter the method of obtaining the equations of motion, the discussion of such trim conditions will be deferred until later.

Application of Lagrange's Method of Undetermined Multipliers to the Case of Complete Rotational Freedom of the Wing Tips

Lagrange's method of undetermined multipliers provides a convenient means of taking account of the constraints in the present problem. This method is based on d'Alembert's Principle, which is an extension of the principle of virtual work to dynamics. For the simple case of the motion of a set of n particles, d'Alembert's Principle states that, for "virtual" displacements δq_i,

$$\sum_{i=1}^{3n} \left(Q_i - m_i\ddot{q}_i\right)\delta q_i = 0$$

where m_i is the mass of one of the particles, q_i is the displacement in the ith degree of freedom, and Q_i is the corresponding applied force. In the present application, let E_i indicate the sum of the inertial, aerodynamic, and weight terms in the unconstrained equation of motion corresponding to a given degree of freedom of an airplane, let F_i indicate any additional applied force in that degree of freedom, and let q_i indicate the displacement in that degree of freedom. Then d'Alembert's Principle takes the form

$$-\sum_{i=1}^{18}\left(F_i - E_i\right)dq_i \equiv \left(-F_Y + m\dot{v} - Y_v v + mV_o\dot{\psi} - \psi W \sin \gamma_o - \phi W \cos \gamma_o\right)\delta y -$$

$$\left(N + N_v v - I_Z\ddot{\psi} + N_{\dot{\psi}}\dot{\psi} + I_{XZ}\ddot{\phi} + N_{\dot{\phi}}\dot{\phi}\right)\delta\psi + \ . \ . \ . = 0 \tag{3}$$

Here $\delta y \equiv \delta q_1$, $\delta\psi \equiv \delta q_2$, and so forth for all eighteen degrees of freedom; and the factor of each δq_i is the equation of motion in the corresponding degree of freedom, for the unconstrained airplanes.

Lagrange's method then requires the conditions relating the virtual displacements in the various degrees of freedom. These equations of condition can be obtained from equations (2). For example, if equation (2.2) is multiplied by δt,

$$\delta y_1 = \delta y + V_o\psi\ \delta t + X\ \delta\psi - Z\ \delta\phi - V_o\psi_1\ \delta t - X_f\ \delta\psi_1 + Z_f\ \delta\phi_1$$

Since the constraint is geometrical, δt may be taken as zero (see reference 6, p. 58). With $\delta t = 0$, each of the six equations of condition obtained from equations (2) is multiplied by an arbitrary parameter λ_k $(k = 1, 2, \ . \ . \ . \ 6)$ and the following equations result:

$$\lambda_1\left(\delta x + Z\ \delta\theta + \frac{b}{2}\ \delta\psi - \delta x_1 - Z_f\ \delta\theta_1 + \frac{b_f}{2}\ \delta\psi_1\right) = 0 \tag{4.1}$$

$$\lambda_2\left(\delta y + X\ \delta\psi - Z\ \delta\phi - \delta y_1 - X_f\ \delta\psi_1 + Z_f\ \delta\phi_1\right) = 0 \tag{4.2}$$

$$\lambda_3\left(\delta z - \frac{b}{2}\ \delta\phi - X\ \delta\theta - \delta z_1 - \frac{b_f}{2}\ \delta\phi_1 + X_f\ \delta\theta_1\right) = 0 \tag{4.3}$$

$$\lambda_4\left(\delta x + Z\ \delta\theta - \frac{b}{2}\ \delta\psi - \delta x_2 - Z_f\ \delta\theta_2 - \frac{b_f}{2}\ \delta\psi_2\right) = 0 \qquad (4.4)$$

$$\lambda_5\left(\delta y + X\ \delta\psi - Z\ \delta\phi - \delta y_2 - X_f\ \delta\psi_2 + Z_f\ \delta\phi_2\right) = 0 \qquad (4.5)$$

$$\lambda_6\left(\delta z + \frac{b}{2}\ \delta\phi - X\ \delta\theta - \delta z_2 + \frac{b_f}{2}\ \delta\phi_2 + X_f\ \delta\theta_2\right) = 0 \qquad (4.6)$$

Since these equations of constraint are integrable, six of the eighteen degrees of freedom can be eliminated by using them (reference 6). It might be noted that the exact equations of constraint, including the trigonometric functions of the variable deflection angles, would not only lead to nonlinear equations of motion but would also lead to nonintegrable constraints. In this case none of the variables could be eliminated and the problem would be extremely complicated.

Equations (4) are now added to equation (3), and the factors of each δq_i are collected. The arbitrary λ_k may be chosen such that six of these factors should vanish, since there are six parameters λ_k. Since there are twelve independent variables in the system, the variables associated with the remaining twelve factors may be considered to be independent and the associated displacements δq_i are therefore arbitrary. Since the displacements δq_i are arbitrary, the factors of the remaining twelve displacements δq_i must also vanish to satisfy the equation obtained by adding equations (4) to equation (3). Thus the parameters λ_k may be chosen such that the factor of each of the eighteen displacements δq_i must vanish and eighteen equations are obtained. For example, the factor of δy from equation (3) is $\left(-F_Y + m\dot{v} - Y_v v + mV_o\dot{\psi} - \psi W \sin\gamma_o - \phi W \cos\gamma_o\right)$, whereas from the sum of equations (4) the factor of δy is $\lambda_2 + \lambda_5$. Setting the sum of these factors equal to zero yields the first equation. The eighteen equations obtained in this fashion are the following:

$$m\dot{v} - Y_v v + mV_o\dot{\psi} - \psi W \sin\gamma_o - \phi W \cos\gamma_o + \lambda_2 + \lambda_5 = F_Y \qquad (5.1)$$

$$-N_v v + I_Z\ddot{\psi} - N_{\dot{\psi}}\dot{\psi} - I_{XZ}\ddot{\phi} - N_{\dot{\phi}}\dot{\phi} + \frac{b}{2}\left(\lambda_1 - \lambda_4\right) + X\left(\lambda_2 + \lambda_5\right) = N \qquad (5.2)$$

$$-L_v v - I_{XZ}\ddot{\psi} - L_{\dot{\psi}}\dot{\psi} + I_X\ddot{\phi} - L_{\dot{\phi}}\dot{\phi} - Z\left(\lambda_2 + \lambda_5\right) + \frac{b}{2}\left(\lambda_6 - \lambda_3\right) = L \qquad (5.3)$$

$$m\dot{u} - X_u u - X_w w + \theta W \cos \gamma_o + \lambda_1 + \lambda_4 = F_X \qquad (5.4)$$

$$-Z_u u + m\dot{w} - Z_w w - \left(Z_{\dot{\theta}} + mV_o\right)\dot{\theta} + \theta W \sin \gamma_o + \lambda_3 + \lambda_6 = F_Z \qquad (5.5)$$

$$-M_u u - M_w w - M_{\dot{w}}\dot{w} - M_{\dot{\theta}}\dot{\theta} + I_Y\ddot{\theta} + Z\left(\lambda_1 + \lambda_4\right) - X\left(\lambda_3 + \lambda_6\right) = M \qquad (5.6)$$

$$m_f\dot{v}_1 - Y_{v_f}v_1 + m_f V_o\dot{\psi}_1 - \psi_1 W_f \sin \gamma_o - \phi_1 W_f \cos \gamma_o - \lambda_2 = F_{Y_1} \qquad (5.7)$$

$$-N_{v_f}v_1 + I_{Z_f}\ddot{\psi}_1 - N_{\dot{\psi}_f}\dot{\psi}_1 - I_{XZ_f}\ddot{\phi}_1 - N_{\dot{\phi}_f}\dot{\phi}_1 + \frac{b_f}{2}\lambda_1 - X_f\lambda_2 = N_1 \qquad (5.8)$$

$$-L_{v_f}v_1 - I_{XZ_f}\ddot{\psi}_1 - L_{\dot{\psi}_f}\dot{\psi}_1 + I_{X_f}\ddot{\phi}_1 - L_{\dot{\phi}_f}\dot{\phi}_1 + Z_f\lambda_2 - \frac{b_f}{2}\lambda_3 = L_1 \qquad (5.9)$$

$$m_f\dot{u}_1 - X_{u_f}u_1 - X_{w_f}w_1 + \theta_1 W_f \cos \gamma_o - \lambda_1 = F_{X_1} \qquad (5.10)$$

$$-Z_{u_f}u_1 + m_f\dot{w}_1 - Z_{w_f}w_1 - \left(Z_{\dot{\theta}_f} + m_f V_o\right)\dot{\theta}_1 + \theta_1 W_f \sin \gamma_o - \lambda_3 = F_{Z_1} \qquad (5.11)$$

$$-M_{u_f}u_1 - M_{w_f}w_1 - M_{\dot{w}_f}\dot{w}_1 - M_{\dot{\theta}_f}\dot{\theta}_1 + I_{Y_f}\ddot{\theta}_1 - Z_f\lambda_1 + X_f\lambda_3 = M_1 \qquad (5.12)$$

$$m_f\dot{v}_2 - Y_{v_f}v_2 + m_fV_o\dot{\psi}_2 - \psi_2W_f \sin \gamma_o - \phi_2W_f \cos \gamma_o - \lambda_5 = F_{Y_2} \tag{5.13}$$

$$-N_{v_f}v_2 + I_{Z_f}\ddot{\psi}_2 - N_{\dot{\psi}_f}\dot{\psi}_2 - I_{XZ_f}\ddot{\phi}_2 - N_{\dot{\phi}_f}\dot{\phi}_2 - \frac{b_f}{2}\lambda_4 - X_f\lambda_5 = N_2 \tag{5.14}$$

$$-L_{v_f}v_2 - I_{XZ_f}\ddot{\psi}_2 - L_{\dot{\psi}_f}\dot{\psi}_2 + I_{X_f}\ddot{\phi}_2 - L_{\dot{\phi}_f}\dot{\phi}_2 + Z_f\lambda_5 + \frac{b_f}{2}\lambda_6 = L_2 \tag{5.15}$$

$$m_f\dot{u}_2 - X_{u_f}u_2 - X_{w_f}w_2 + \theta_2W_f \cos \gamma_o - \lambda_4 = F_{X_2} \tag{5.16}$$

$$-Z_{u_f}u_2 + m_f\dot{w}_2 - Z_{w_f}w_2 - \left(Z_{\dot{\theta}_f} + m_fV_o\right)\dot{\theta}_2 + \theta_2W_f \sin \gamma_o - \lambda_6 = F_{Z_2} \tag{5.17}$$

$$-M_{u_f}u_2 - M_{w_f}w_2 - M_{\dot{w}_f}\dot{w}_2 - M_{\dot{\theta}_f}\dot{\theta}_2 + I_{Y_f}\ddot{\theta}_2 - Z_f\lambda_4 + X_f\lambda_6 = M_2 \tag{5.18}$$

In these equations the terms in the undetermined parameters represent the constraining forces arising from the wing-tip connections. Equations (4) and (5) now give twenty-four equations in the twenty-four unknowns consisting of the eighteen degrees of freedom and the six parameters λ_k. Moreover, since the constraints given by equations (4) are integrable, the system can be completely described by only twelve independent degrees of freedom. The twelve independent equations of motion can be obtained by first solving for the constraint parameters λ_k and then substituting these values into any twelve of equations (5). Any six of the original degrees of freedom may now be eliminated by using the constraint conditions in the convenient form given by equations (2).

The most convenient method of carrying out this process is to obtain twelve linearly independent combinations of equations (5) which eliminate the parameters λ_k. This may be done in many equivalent ways, but the following twelve linearly independent combinations seem to be the simplest:

Side-force equation:	$(5.1) + (5.7) + (5.13)$	(6.1)
Longitudinal-force equation:	$(5.4) + (5.10) + (5.16)$	(6.2)
Normal-force equation:	$(5.5) + (5.11) + (5.17)$	(6.3)
Bomber yaw equation:	$(5.2) - X(5.1) + \frac{b}{2}\left[(5.10) - (5.16)\right]$	(6.4)
Bomber roll equation:	$(5.3) + Z(5.1) + \frac{b}{2}\left[(5.17) - (5.11)\right]$	(6.5)
Bomber pitch equation:	$(5.6) + X(5.5) - Z(5.4)$	(6.6)
Left-fighter yaw equation:	$(5.8) + \frac{b_f}{2}(5.10) - X_f(5.7)$	(6.7)
Left-fighter roll equation:	$(5.9) + Z_f(5.7) - \frac{b_f}{2}(5.11)$	(6.8)
Left-fighter pitch equation:	$(5.12) - Z_f(5.10) + X_f(5.11)$	(6.9)
Right-fighter yaw equation:	$(5.14) - \frac{b_f}{2}(5.16) - X_f(5.13)$	(6.10)
Right-fighter roll equation:	$(5.15) + Z_f(5.13) + \frac{b_f}{2}(5.17)$	(6.11)
Right-fighter pitch equation:	$(5.18) - Z_f(5.16) + X_f(5.17)$	(6.12)

The six translational degrees of freedom of the fighters may be eliminated from equations (6) by using equations (2). Then these are the twelve linearly independent differential equations in the twelve remaining independent degrees of freedom which determine the small motions of the configuration. Any other combination of equations (5) which eliminates the parameters λ_k may be written as a linear combination of these twelve equations. The first three equations give the combined translational forces on the system, the next three equations give the rotational moments on the bomber, and the final six equations give the rotational moments on the fighters. The solution of this set of equations would give the translational motion of the bomber center of gravity and the rotational motion of the bomber and both fighters. The translational motion of the fighters is of little interest, but may, of course, be obtained from equation (2).

SYMMETRIZATION OF EQUATIONS OF MOTION

The equations presented are not in the proper form to show the symmetry of the system. The proper form can be obtained by expressing the fighter motions and forces in symmetrical and antisymmetrical components. For example, consider the rolling equations of the fighters. The rolling equation of the left fighter, equation (6.8), is

$$
\begin{aligned}
& m_f Z_f \dot{v} - \left(L_{v_f} + Z_f Y_{v_f}\right) v + m_f X Z_f \ddot{\psi} + \left[m_f Z_f V_o - X\left(L_{v_f} + Z_f Y_{v_f}\right) + \frac{b b_f}{4} Z_{u_f}\right] \dot{\psi} - \\
& \left(L_{v_f} + Z_f Y_{v_f}\right) V_o \psi + m_f\left(\frac{b b_f}{4} - Z Z_f\right) \ddot{\phi} + \left[Z\left(L_{v_f} + Z_f Y_{v_f}\right) - \frac{b b_f}{4} Z_{w_f}\right] \dot{\phi} + \\
& \frac{b_f}{2} Z_{u_f} u - m_f \frac{b_f}{2} \dot{w} + \frac{b_f}{2} Z_{w_f} w + m_f X \frac{b_f}{2} \ddot{\theta} + \frac{b_f}{2}\left(m_f V_o + Z Z_{u_f} - X Z_{w_f}\right) \dot{\theta} - \\
& \frac{b_f}{2} Z_{w_f} V_o \theta + \left[I_{X_f} + m_f\left(\frac{b_f^2}{4} + Z_f^2\right)\right] \ddot{\phi}_1 - \Bigg[L_{\dot{\phi}_f} + Z_f\left(L_{v_f} + Z_f Y_{v_f}\right) + \\
& \frac{b_f^2}{4} Z_{w_f}\Bigg] \dot{\phi}_1 - \phi_1 Z_f W_f \cos \gamma_o - m_f X_f \frac{b_f}{2} \ddot{\theta}_1 + \frac{b_f}{2}\left(Z_{\dot{\theta}_f} + X_f Z_{w_f} - Z_f Z_{u_f}\right) \dot{\theta}_1 + \\
& \frac{b_f}{2}\left(V_o Z_{w_f} - W_f \sin \gamma_o\right) \theta_1 - \left(I_{XZ_f} + m_f X_f Z_f\right) \ddot{\psi}_1 + \Bigg[X_f\left(L_{v_f} + Z_f Y_{v_f}\right) - L_{\dot{\psi}_f} + \\
& \frac{b_f^2}{4} Z_{u_f}\Bigg] \dot{\psi}_1 + \left[\left(L_{v_f} + Z_f Y_{v_f}\right) V_o - Z_f W_f \sin \gamma_o\right] \psi_1 = L_1 + Z_f F_{Y_1} - \frac{b_f}{2} F_{Z_1}
\end{aligned}
\tag{7}
$$

The rolling equation of the right fighter, equation (6.11), is the same as equation (7) except that the subscript 2 replaces the subscript 1, $-b$ replaces b, and $-b_f$ replaces b_f; that is,

$$
\begin{aligned}
&m_fZ_f\dot{v} - \left(L_{v_f} + Z_fY_{v_f}\right)v + m_fXZ_f\ddot{\psi} + \left[m_fZ_fV_o - X\left(L_{v_f} + Z_fY_{v_f}\right) + \frac{bb_f}{4}Z_{u_f}\right]\dot{\psi} - \\
&\left(L_{v_f} + Z_fY_{v_f}\right)V_o\psi + m_f\left(\frac{bb_f}{4} - ZZ_f\right)\ddot{\phi} + \left[Z\left(L_{v_f} + Z_fY_{v_f}\right) - \frac{bb_f}{4}Z_{w_f}\right]\dot{\phi} - \\
&\frac{b_f}{2}Z_{u_f}u + m_f\frac{b_f}{2}\dot{w} - \frac{b_f}{2}Z_{w_f}w - m_fX\frac{b_f}{2}\ddot{\theta} - \frac{b_f}{2}\left(m_fV_o + ZZ_{u_f} - XZ_{w_f}\right)\dot{\theta} + \\
&\frac{b_f}{2}Z_{w_f}V_o\theta + \left[I_{X_f} + m_f\left(\frac{b_f^2}{4} + Z_f^2\right)\right]\ddot{\phi}_2 - \left[L_{\dot{\phi}_f} + Z_f\left(L_{v_f} + Z_fY_{v_f}\right) + \right. \\
&\left.\frac{b_f^2}{4}Z_{w_f}\right]\dot{\phi}_2 - \phi_2Z_fW_f\cos\gamma_o + m_fX_f\frac{b_f}{2}\ddot{\theta}_2 - \frac{b_f}{2}\left(Z_{\dot{\theta}_f} + X_fZ_{w_f} - Z_fZ_{u_f}\right)\dot{\theta}_2 - \\
&\frac{b_f}{2}\left(V_oZ_{w_f} - W_f\sin\gamma_o\right)\theta_2 - \left(I_{XZ_f} + m_fX_fZ_f\right)\ddot{\psi}_2 + \left[X_f\left(L_{v_f} + Z_fY_{v_f}\right) - L_{\dot{\psi}_f} + \right. \\
&\left.\frac{b_f^2}{4}Z_{u_f}\right]\dot{\psi}_2 + \left[\left(L_{v_f} + Z_fY_{v_f}\right)V_o - Z_fW_f\sin\gamma_o\right]\psi_2 = L_2 + Z_fF_{Y_2} + \frac{b_f}{2}F_{Z_2}
\end{aligned}
\tag{8}
$$

These equations do not show the symmetry of the system, since they contain both longitudinal and lateral degrees of freedom. These two equations, however, may be replaced by their sum and their difference; physically, this substitution is equivalent to replacing the individual fighter rolling moments by the equivalent symmetrical and antisymmetrical components of the combined fighter rolling moments. The symmetrical and antisymmetrical fighter rotations may be defined as follows:

$$\phi_s \equiv \frac{1}{2}\left(\phi_1 - \phi_2\right) \qquad \theta_s \equiv \frac{1}{2}\left(\theta_1 + \theta_2\right) \qquad \psi_s \equiv \frac{1}{2}\left(\psi_1 - \psi_2\right) \tag{9}$$

$$\phi_a \equiv \frac{1}{2}\left(\phi_1 + \phi_2\right) \qquad \theta_a \equiv \frac{1}{2}\left(\theta_1 - \theta_2\right) \qquad \psi_a \equiv \frac{1}{2}\left(\psi_1 + \psi_2\right) \tag{10}$$

Then, one-half the sum of equations (7) and (8) is

$$
\begin{aligned}
& m_f Z_f \dot{v} - \left(L_{v_f} + Z_f Y_{v_f}\right) v + m_f X Z_f \ddot{\psi} + \left[m_f Z_f V_o - X\left(L_{v_f} + Z_f Y_{v_f}\right) + \frac{bb_f}{4} Z_{u_f}\right] \dot{\psi} - \\
& \quad \left(L_{v_f} + Z_f Y_{v_f}\right) V_o \psi + m_f\left(\frac{bb_f}{4} - ZZ_f\right)\ddot{\phi} + \left[Z\left(L_{v_f} + Z_f Y_{v_f}\right) - \frac{bb_f}{4} Z_{w_f}\right]\dot{\phi} + \\
& \quad \left[I_{X_f} + m_f\left(\frac{{b_f}^2}{4} + {Z_f}^2\right)\right]\ddot{\phi}_a - \left[L_{\dot{\phi}_f} + Z_f\left(L_{v_f} + Z_f Y_{v_f}\right) + \frac{{b_f}^2}{4} Z_{w_f}\right]\dot{\phi}_a - \\
& \quad \phi_a Z_f W_f \cos\gamma_o - m_f X_f \frac{b_f}{2}\ddot{\theta}_a + \frac{b_f}{2}\left(Z_{\dot{\theta}_f} + X_f Z_{w_f} - Z_f Z_{u_f}\right)\dot{\theta}_a + \frac{b_f}{2}\Big(V_o Z_{w_f} - \\
& \quad W_f \sin\gamma_o\Big)\theta_a - \left(I_{XZ_f} + m_f X_f Z_f\right)\ddot{\psi}_a + \left[X_f\left(L_{v_f} + Z_f Y_{v_f}\right) - L_{\dot{\psi}_f} + \frac{{b_f}^2}{4} Z_{u_f}\right]\dot{\psi}_a + \\
& \quad \left[\left(L_{v_f} + Z_f Y_{v_f}\right)V_o - Z_f W_f \sin\gamma_o\right]\psi_a = \frac{1}{2}\Big[L_1 + L_2 + Z_f\left(F_{Y_1} + F_{Y_2}\right) - \\
& \quad \frac{b_f}{2}\left(F_{Z_1} - F_{Z_2}\right)\Big] \qquad (11)
\end{aligned}
$$

Also, one-half the difference of equations (7) and (8) is

$$
\begin{aligned}
& \frac{b_f}{2} Z_{u_f} u - m_f \frac{b_f}{2} \dot{w} + \frac{b_f}{2} Z_{w_f} w + m_f X \frac{b_f}{2} \ddot{\theta} + \frac{b_f}{2}\left(m_f V_o + Z Z_{u_f} - X Z_{w_f}\right)\dot{\theta} - \\
& \quad \frac{b_f}{2} Z_{w_f} V_o \theta + \left[I_{X_f} + m_f\left(\frac{{b_f}^2}{4} + {Z_f}^2\right)\right]\ddot{\phi}_s - \Big[L_{\dot{\phi}_f} + Z_f\left(L_{v_f} + Z_f Y_{v_f}\right) + \\
& \quad \frac{{b_f}^2}{4} Z_{w_f}\Big]\dot{\phi}_s - \phi_a Z_f W_f \cos\gamma_o - m_f X_f \frac{b_f}{2}\ddot{\theta}_s + \frac{b_f}{2}\left(Z_{\dot{\theta}_f} + X_f Z_{w_f} - Z_f Z_{u_f}\right)\dot{\theta}_s + \\
& \quad \frac{b_f}{2}\left(V_o Z_{w_f} - W_f \sin\gamma_o\right)\theta_s - \left(I_{XZ_f} + m_f X_f Z_f\right)\ddot{\psi}_s + \Big[X_f\left(L_{v_f} + Z_f Y_{v_f}\right) - L_{\dot{\psi}_f} + \\
& \quad \frac{{b_f}^2}{4} Z_{u_f}\Big]\dot{\psi}_s + \left[\left(L_{v_f} + Z_f Y_{v_f}\right)V_o - Z_f W_f \sin\gamma_o\right]\psi_s = \frac{1}{2}\Big[L_1 - L_2 + \\
& \quad Z_f\left(F_{Y_1} - F_{Y_2}\right) - \frac{b_f}{2}\left(F_{Z_1} + F_{Z_2}\right)\Big] \qquad (12)
\end{aligned}
$$

The variables occurring in equation (11) are v, ψ, ϕ, ϕ_a, θ_a, and ψ_a, while those occurring in equation (12) are u, w, θ, ϕ_s, θ_s, and ψ_s. The set of variables occurring in equation (11) may be designated as the lateral degrees of freedom of the entire system and those occurring in equation (12) may be designated as the longitudinal degrees of freedom. That is, the lateral degrees of freedom of the system are the ordinary lateral motions of the bomber and the antisymmetrical components of the combined fighter motions as defined in equations (10). The longitudinal degrees of freedom are the longitudinal bomber motions and the symmetrical components of the fighter motions as defined in equations (9).

If the remaining two pairs of fighter equations (equations (6.7) and (6.10), and equations (6.9) and (6.12)) are similarly replaced by one-half their sums and differences, by making the substitutions given in equations (9) and (10), the same separation of variables occurs. Finally, using equations (9) and (10) in the six bomber equations completes the separation of the twelve equations of the entire system into six lateral equations and six longitudinal equations, which are independent of each other. These equations are presented in appendix A.

The longitudinal and lateral equations may now be treated separately. The stability of the longitudinal motions may be determined by the well-known method of expanding the determinant of the longitudinal equations and evaluating the roots of the resulting characteristic equation. The same may of course be done independently for the lateral motion. Actual solutions may be obtained by any of the well-known methods for solving sets of linear differential equations, such as the Laplace transform method, some step-by-step method, or by use of an analog calculator. If the rotational motion of a given fighter is desired, it is simply necessary to use equations (9) and (10). For example, $\phi_1 = \phi_s + \phi_a$, and $\phi_2 = \phi_a - \phi_s$. In order to obtain the individual fighter motions, both the lateral and longitudinal equations must be solved, since these motions contain both the lateral and longitudinal modes of motion.

These equations hold for the most general type of coupling likely to be encountered. In practical cases many simplifications will probably be possible. For example, Z and Z_f will generally be much smaller than $\frac{b}{2}$ and $\frac{b_f}{2}$, respectively. Also, for unswept wings X and X_f will be much smaller than $\frac{b}{2}$ and $\frac{b_f}{2}$, respectively. These facts in conjunction with the approximate magnitudes of the familiar stability derivatives of the individual airplanes show that it may be possible, in practical cases, to ignore many terms in these equations.

Note that the set of lateral equations is eleventh order in the time-derivative operator and the longitudinal equations are tenth order. Also, neither characteristic equation has a zero root in the general case. The solution of these most general equations of motion may, therefore, be a formidable task. In order to determine the effect of varying parameters on the motion of the configuration, it would be necessary to use an analog computer. The fighter rotational motions must be restricted in order to reduce the differential order of the problem.

CONSIDERATION OF STEADY-STATE FIGHTER DEFLECTIONS

In deriving the equations of motion it was assumed that there were no steady-state angles between the airplanes in the steady-state condition. Inasmuch as such angles may exist, their effect is now considered.

In the practical case, it seems reasonable to assume that the yawing moment of the fighters due to their drag will be trimmed out so that the steady-state fighter yaw angles vanish. Since the X-axis is arbitrarily chosen along the initial flight path, the initial pitch angle also vanishes. The remaining angle which may be considered is a possible angle of steady-state "droop" of the fighter wings.

In the following analysis, γ_o is assumed to be zero; then the steady-state roll angles are determined by the following equations of trim for side force, vertical force, and fighter roll about the left and right coupling joints, respectively:

$$L'_f\left[\sin(\phi_1)_o + \sin(\phi_2)_o\right] = 0$$

$$L' - W + L'_f\left[\cos(\phi_1)_o + \cos(\phi_2)_o\right] - 2W_f = 0$$

$$\frac{b_f}{2}\left[L'_f - W_f \cos(\phi_1)_o\right] = 0$$

$$\frac{b_f}{2}\left[L'_f - W_f \cos(\phi_2)_o\right] = 0$$

The first of these equations imposes the obvious condition that $(\phi_1)_o = -(\phi_2)_o$; that is, $(\phi_a)_o = 0$. Let $(\phi_2)_o \equiv \phi_o$, then $(\phi_1)_o = -\phi_o$, and the last two equations are identical. The remaining conditions then are

$$L'_f = W_f \cos \phi_o \tag{13}$$

$$L' = W + 2W_f\left(1 - \cos^2\phi_o\right) \tag{14}$$

These equations show that the configuration may be trimmed with the lift on each fighter not sufficient to support its own weight, in which case the trim angle of droop will be $\phi_o = \cos^{-1} \frac{L'_f}{W_f}$. Equation (14) also shows that the bomber lift will have to be correspondingly increased to carry the unbalanced weight.

Note that the trim equations are satisfied for negative droop also, since ϕ_o appears only in the cosine function. This fact implies that there is a trim condition with the fighter wings poised above the bomber wing level. However, a simple consideration of the effect of slight variations of fighter lift or roll angle on the fighter rolling moment about the hinge shows that the upper trim position is statically unstable, the $\phi_o = 0$ position is a position of neutral equilibirum, and the lower trim position is statically stable. It might therefore be desirable to allow the fighter wings to droop somewhat if the bomber wings are capable of sustaining the additional steady-state load. It should be noted that the aerodynamic interference effects have been ignored in this discussion. These effects might make positions of negative droop stable.

The case of steady-state fighter roll angles, such as have just been described, may be handled by making the appropriate change in equations (2.2), (2.3), (2.5), and (2.6). Equation (1) now implies, when $(\phi_2)_o = -(\phi_1)_o = \phi_o$, that

$$v + V_o\psi + X\dot{\psi} - Z\dot{\phi} = \left(v_1 + V_o\psi_1 + X_f\dot{\psi}_1 - Z_f\dot{\phi}_1\right)\cos \phi_o + \left(w_1 - V_o\theta_1 + \frac{b_f}{2}\dot{\phi}_1 - X_f\dot{\theta}_1\right)\sin \phi_o$$

$$v + V_o\psi + X\dot{\psi} - Z\dot{\phi} = \left(v_2 + V_o\psi_2 + X_f\dot{\psi}_2 - Z_f\dot{\phi}_2\right)\cos\phi_o -$$
$$\left(w_2 - V_o\theta_2 - \frac{b_f}{2}\dot{\phi}_2 - X_f\dot{\theta}_2\right)\sin\phi_o$$

$$w - V_o\theta - \frac{b}{2}\dot{\phi} - X\dot{\theta} = \left(w_1 - V_o\theta_1 + \frac{b_f}{2}\dot{\phi}_1 - X_f\dot{\theta}_1\right)\cos\phi_o -$$
$$\left(v_1 + V_o\psi_1 + X_f\dot{\psi}_1 - Z_f\dot{\phi}_1\right)\sin\phi_o$$

$$w - V_o\theta + \frac{b}{2}\dot{\phi} - X\dot{\theta} = \left(w_2 - V_o\theta_2 - \frac{b_f}{2}\dot{\phi}_2 - X_f\dot{\theta}_2\right)\cos\phi_o +$$
$$\left(v_2 + V_o\psi_2 + X_f\dot{\psi}_2 - Z_f\dot{\phi}_2\right)\sin\phi_o$$

These equations give the modified equations of condition and replace equations (2.2), (2.3), (2.5), and (2.6) as follows:

$$v_1 = -V_o\psi_1 - X_f\dot{\psi}_1 + Z_f\dot{\phi}_1 + \left(v + V_o\psi + X\dot{\psi} - Z\dot{\phi}\right)\cos\phi_o -$$
$$\left(w - V_o\theta - \frac{b}{2}\dot{\phi} - X\dot{\theta}\right)\sin\phi_o \tag{15.1}$$

$$v_2 = -V_o\psi_2 - X_f\dot{\psi}_2 + Z_f\dot{\phi}_2 + \left(v + V_o\psi + X\dot{\psi} - Z\dot{\phi}\right)\cos\phi_o +$$
$$\left(w - V_o\theta + \frac{b}{2}\dot{\phi} - X\dot{\theta}\right)\sin\phi \tag{15.2}$$

$$w_1 = V_o\theta_1 - \frac{b_f}{2}\dot{\phi}_1 + X_f\dot{\theta}_1 + \left(w - V_o\theta - \frac{b}{2}\dot{\phi} - X\dot{\theta}\right)\cos\phi_o +$$
$$\left(v + V_o\psi + X\dot{\psi} - Z\dot{\phi}\right)\sin\phi_o \tag{15.3}$$

$$w_2 = V_o\theta_2 + \frac{b_f}{2}\dot{\phi}_2 + X_f\dot{\theta}_2 + \left(w - V_o\theta + \frac{b}{2}\dot{\phi} - X\dot{\theta}\right)\cos\phi_o -$$

$$\left(v + V_o\psi + X\dot{\psi} - Z\dot{\phi}\right)\sin\phi_o \tag{15.4}$$

It is easily seen that these conditions will not destroy the symmetry properties of the final equations of motion. Actually, the most logical way to derive the equations of motion in symmetrical form would be to use the equations of condition in symmetrical form (sum and difference equations, with symmetrical and antisymmetrical fighter variables), together with the independent fighter equations in symmetrical form replacing the usual independent fighter equations given in equations (5). This method was not used only because it was not desired to confuse unnecessarily the discussion of the application of Lagrange's method by introducing questions of symmetry.

In symmetrical form, equations (15) become

$$v_a \equiv \frac{v_1 + v_2}{2} = -V_o\psi_a - X_f\dot{\psi}_a + Z_f\dot{\phi}_a + \left(v + V_o\psi + X\dot{\psi} - Z\dot{\phi}\right)\cos\phi_o + b\dot{\phi}\sin\phi_o \tag{16.1}$$

$$v_s \equiv \frac{v_1 - v_2}{2} = -V_o\psi_s - X_f\dot{\psi}_s + Z_f\dot{\phi}_s - \left(w - V_o\theta - X\dot{\theta}\right)\sin\phi_o \tag{16.2}$$

$$w_s \equiv \frac{w_1 + w_2}{2} = V_o\theta_s - \frac{b_f}{2}\dot{\phi}_s + X_f\dot{\theta}_s + \left(w - V_o\theta - X\dot{\theta}\right)\cos\phi_o \tag{16.3}$$

$$w_a \equiv \frac{w_1 - w_2}{2} = V_o\theta_a - \frac{b_f}{2}\dot{\phi}_a + X_f\dot{\theta}_a - b\dot{\phi}\cos\phi_o + \left(v + V_o\psi + X\dot{\psi} - Z\dot{\phi}\right)\sin\phi_o \tag{16.4}$$

It can be seen that v_a and w_a, which will enter into the lateral equations of motion, contain only lateral variables, whereas w_s and v_s, which enter only into the longitudinal equations, contain only longitudinal variables. Therefore, the symmetry properties of the final equations are preserved even when steady-state roll angles are assumed. Physically it is clear that any steady-state angles which preserve the symmetry of the steady-state configuration will give rise to separable lateral and longitudinal equations.

SIMPLIFIED EQUATIONS FOR MORE RESTRICTED COUPLING

Some of the possible simplifications of the general equations of motion given in appendix A have already been mentioned. For example, if any of the coupling-joint position components $(X,\ \ Z,\ \ X_f,$ and $Z_f)$ should vanish, many terms in the general equations would not appear. A more fundamental simplification is that in which the rotation at the coupling joints is restricted to less than three degrees of freedom. In such cases additional equations of condition may be imposed; two of the variables are thereby eliminated for each restriction in a component of the rotational motion. Thus, the number of equations and the differential order of the equations of motion which must finally be solved are reduced.

If the fighters are restricted, by the design of the coupling, to rotate about only one or two axes instead of being completely free to rotate, then the equations of condition will generally involve the direction cosines of the hinge axes with respect to the steady-state axes. The method of treating these cases may best be illustrated by carrying out the case for rotation about one axis only, corresponding to an ordinary hinge coupling.

The usual practical type of coupling is a hinge type of connection, the axis of which is parallel to the symmetry plane of the bomber. If the hinge axis is parallel to the XZ-plane, the fighters and bomber are rigidly connected in pitch, and two additional equations of condition may immediately be written as

$$\theta_1 = \theta \tag{17.1}$$

$$\theta_2 = \theta \tag{17.2}$$

Now if the X-axes in the individual airplanes may be chosen such that the hinge axis is in the direction common to the steady-state X-axes, then the additional two equations of restriction are simply

$$\psi_1 = \psi \tag{17.3}$$

$$\psi_2 = \psi \tag{17.4}$$

In general, such a choice of the steady-state X-axes would only be permissible if body axes were used instead of stability axes. In this case the original equations of motion represented by E_1 in equation (3) would have to be written in terms of body axes. However, if stability axes are to be used in setting up the uncoupled equations of motion, as has been done in equations (5), then the direction of the steady-state X-axes must be along the steady-state velocity. If the hinge axis is assumed to be fixed in the bomber or in the fighter wings, some steady-state angle will generally exist between the hinge axis and the velocity direction. This angle will be called η_o and is assumed positive when the hinge is pitched above the velocity in the steady state. Let the disturbance deflection of the fighter about the hinge axis be called σ and be taken positive in the same direction as positive roll. Since the disturbance angles are small they may be treated as vectors, and the components of σ_f along the steady-state X- and Z-axes are $\phi_f - \phi$ and $\psi_f - \psi$, respectively. Therefore,

$$\phi_1 - \phi = \sigma_1 \cos \eta_o \qquad \phi_2 - \phi = \sigma_2 \cos \eta_o$$

$$\psi_1 - \psi = -\sigma_1 \sin \eta_o \qquad \psi_2 - \psi = -\sigma_2 \sin \eta_o$$

Eliminating σ_1 and σ_2 from these equations yields

$$\left(\phi_1 - \phi\right)\sin \eta_o = \left(\psi - \psi_1\right)\cos \eta_o$$

$$\left(\phi_2 - \phi\right)\sin \eta_o = \left(\psi - \psi_2\right)\cos \eta_o$$

Thus, the equations of condition corresponding to equations (17.3) and (17.4) are

$$\psi_1 = \psi + \left(\phi - \phi_1\right)\tan \eta_o \qquad (17.3')$$

$$\psi_2 = \psi + \left(\phi - \phi_2\right)\tan \eta_o \qquad (17.4')$$

In this case the hinge has been assumed to lie parallel to the XZ-plane. By using equations (17) the four variables θ_f and ψ_f may be eliminated. In the most general case of hinge coupling, the hinge axis will not necessarily be perpendicular to the Y-axis, but will make certain steady-state angles with all the axes. Let these angles be η_1, η_2, and η_3 with respect to the X-, Y-, and Z-axis, respectively. Then by proceeding exactly as before, equations (17) become

$$\theta_1 = \theta + \left(\phi_1 - \phi\right)\frac{\cos\eta_2}{\cos\eta_1} \tag{18.1}$$

$$\theta_2 = \theta + \left(\phi_2 - \phi\right)\frac{\cos\eta_2}{\cos\eta_1} \tag{18.2}$$

$$\psi_1 = \psi + \left(\phi_1 - \phi\right)\frac{\cos\eta_3}{\cos\eta_1} \tag{18.3}$$

$$\psi_2 = \psi + \left(\phi_2 - \phi\right)\frac{\cos\eta_3}{\cos\eta_1} \tag{18.4}$$

These are the general equations of condition which must be used along with equations (2) in applying Lagrange's method for hinge coupling when the hinge is skewed with respect to the symmetry plane.

In order to illustrate the modifications introduced by the additional conditions, the simplest conditions, represented by equations (17.1) to (17.4), will be considered. By proceeding as before, the additional equations corresponding to equations (4) are

$$\lambda_7\left(\delta\theta - \delta\theta_1\right) = 0 \tag{19.1}$$

$$\lambda_8\left(\delta\theta - \delta\theta_2\right) = 0 \tag{19.2}$$

$$\lambda_9\left(\delta\psi - \delta\psi_1\right) = 0 \tag{19.3}$$

$$\lambda_{10}\left(\delta\psi - \delta\psi_2\right) = 0 \tag{19.4}$$

These conditions introduce the following additional terms into equations (5): $(\lambda_9 + \lambda_{10})$ in equation (5.2), $(\lambda_7 + \lambda_8)$ in equation (5.6), $-\lambda_9$ in equation (5.8), $-\lambda_7$ in equation (5.12), $-\lambda_{10}$ in equation (5.14), and $-\lambda_8$ in equation (5.18). Six of the equations of motion given in equation (6) will not be changed and will therefore still be valid in this case. These are equations (6.1), (6.2), (6.3), (6.5), (6.8), and (6.11) which are the translational equations and roll equations. The substitutions given by equations (17) make it possible to eliminate four more variables, so that only eight variables remain. Therefore, when equations (17) are used in the six equations which remain unchanged, only two additional independent equations are needed. In fact, only two more linearly independent combinations of equations (5) which eliminate the λ_k are possible. In order to eliminate the additional λ_k introduced, the simplest combinations of equations would seem to be

$$(5.2) + (5.8) + (5.14) + \frac{b + b_f}{2}\left[(5.10) - (5.16)\right] \tag{20.1}$$

$$(5.6) + (5.12) + (5.18) \tag{20.2}$$

The equations of motion may now be symmetrized exactly as before. The nondimensional variables u', β, and α are used in place of u, v, and w, respectively. Then, for the case of hinge coupling with the hinge axis along the steady-state X-axis, if it is also assumed that $X = Z = X_f = Z_f = 0$ in order to obtain the simplest possible case, the equations of lateral motion are

$$(m + 2m_f)V_o\dot{\beta} - \left(Y_\beta + 2Y_{\beta_f}\right)\beta + (m + 2m_f)V_o\dot{\psi} - \psi\left(W + 2W_f\right)\sin\gamma_o - \phi W\cos\gamma_o - 2\phi_a W_f\cos\gamma_o = F_Y + F_{Y_1} + F_{Y_2} \tag{21.1}$$

$$-\left(N_\beta + 2N_{\beta_f}\right)\beta + \left[I_Z + 2I_{Z_f} + \frac{m_f}{2}\left(b + b_f\right)^2\right]\ddot{\psi} - \left[N_{\dot{\psi}} + 2N_{\dot{\psi}_f} + \frac{1}{2V_o}\left(b + b_f\right)^2 X_{u'_f}\right]\dot{\psi} - I_{XZ}\ddot{\phi} + \left[\frac{b\left(b + b_f\right)}{2V_o}X_{\alpha_f} - N_{\dot{\phi}}\right]\dot{\phi} - 2I_{XZ_f}\ddot{\phi}_a + \left[\frac{b_f\left(b + b_f\right)}{2V_o}X_{\alpha_f} - 2N_{\dot{\phi}_f}\right]\dot{\phi}_a = N + N_1 + N_2 + \frac{b + b_f}{2}\left(F_{X_1} - F_{X_2}\right) \tag{21.2}$$

$$-L_{\beta}\beta - I_{XZ}\ddot{\psi} + \left[\frac{b\left(b + b_f\right)}{2V_o} Z_{u'_f} - L_{\dot{\psi}}\right]\dot{\psi} + \left(I_X + m_f \frac{b^2}{2}\right)\ddot{\phi} - \left(L_{\dot{\phi}} + \frac{b^2}{2V_o} Z_{\alpha_f}\right)\dot{\phi} + m_f \frac{bb_f}{2} \ddot{\phi}_a - \frac{bb_f}{2V_o} Z_{\alpha_f}\dot{\phi}_a = L - \frac{b}{2}\left(F_{Z_1} - F_{Z_2}\right) \tag{21.3}$$

$$-L_{\beta_f}\beta - I_{XZ_f}\ddot{\psi} + \left[\frac{b_f\left(b + b_f\right)}{4V_o} Z_{u'_f} - L_{\dot{\psi}_f}\right]\dot{\psi} + m_f \frac{bb_f}{4} \ddot{\phi} - \frac{bb_f}{4V_o} Z_{\alpha_f}\dot{\phi} + \left(I_{X_f} + m_f \frac{b_f^2}{4}\right)\ddot{\phi}_a - \left(L_{\dot{\phi}_f} + \frac{b_f^2}{4V_o} Z_{\alpha_f}\right)\dot{\phi}_a = \frac{1}{2}\left[L_1 + L_2 - \frac{b_f}{2}\left(F_{Z_1} - F_{Z_2}\right)\right] \tag{21.4}$$

and the equations of longitudinal motion are

$$\left(I_Y + 2I_{Y_f}\right)\ddot{\theta} - \left(M_{\dot{\theta}} + 2M_{\dot{\theta}_f}\right)\dot{\theta} - \left(M_{u'} + 2M_{u'_f}\right)u' - \left(M_{\dot{\alpha}} + 2M_{\dot{\alpha}_f}\right)\dot{\alpha} - \left(M_{\alpha} + 2M_{\alpha_f}\right)\alpha + \frac{b_f}{V_o} M_{\dot{\alpha}_f}\ddot{\phi}_s + \frac{b_f}{V_o} M_{\alpha_f}\dot{\phi}_s = M + M_1 + M_2 \tag{21.5}$$

$$\theta\left(W + 2W_f\right)\cos \gamma_o + \left(m + 2m_f\right)V_o\dot{u}' - \left(X_{u'} + 2X_{u'_f}\right)u' - \left(X_{\alpha} + 2X_{\alpha_f}\right)\alpha + \frac{b_f}{V_o} X_{\alpha_f}\dot{\phi}_s = F_X + F_{X_1} + F_{X_2} \tag{21.6}$$

$$-\left[Z_{\dot{\theta}} + 2Z_{\dot{\theta}_f} + \left(m + 2m_f\right)V_o\right]\dot{\theta} + \theta\left(W + 2W_f\right)\sin \gamma_o - \left(Z_{u'} + 2Z_{u'_f}\right)u' + \left(m + 2m_f\right)V_o\dot{\alpha} - \left(Z_{\alpha} + 2Z_{\alpha_f}\right)\alpha - m_f b_f\ddot{\phi}_s + \frac{b_f}{V_o} Z_{\alpha_f}\dot{\phi}_s = F_Z + F_{Z_1} + F_{Z_2} \tag{21.7}$$

$$\frac{b_f}{2}\left(m_f V_o + Z_{\dot{\theta}_f}\right)\dot{\theta} - \theta \frac{b_f}{2} W_f \sin \gamma_o + \frac{b_f}{2} Z_{u'_f} u' - m_f \frac{b_f}{2} V_o\dot{\alpha} + \frac{b_f}{2} Z_{\alpha_f}\alpha + \left(I_{X_f} + \frac{m_f b_f^2}{4}\right)\ddot{\phi}_s - \left(L_{\dot{\phi}_f} + \frac{b_f^2}{4V_o} Z_{\alpha_f}\right)\dot{\phi}_s = \frac{1}{2}\left[L_1 - L_2 - \frac{b_f}{2}\left(F_{Z_1} + F_{Z_2}\right)\right] \tag{21.8}$$

It might be noted that the equations of motion in this case could have been obtained directly from the general case given in equations (6) by simply adding the three pitch equations and the three yaw equations and using $\psi = \psi_1 = \psi_2$ and $\theta = \theta_1 = \theta_2$ to obtain equations (20.1) and (20.2). The reason for this is that the constraints in this case are the same as in the general case except that two additional constraints are added. The method of introducing the additional λ_k was used in order to show how the more general constraints (such as given in equations (18)) might be handled. Also, if the general equations are not known, it is always more convenient to introduce all the constraints as done here, rather than first deriving the general equations.

The system described by equations (21) is probably the simplest wing-tip-coupled system. The lateral equations can be seen to have a seventh-order characteristic equation, but, since there are two zero roots, the problem is essentially fifth order. The longitudinal motion has a sixth-order characteristic equation with one zero root and is therefore also essentially of fifth order.

Clearly, there are many possible cases which are less general than that described by the general equations in appendix A but more general than that described by equations (21). It is hoped that the discussion of the modifications introduced by additional constraints has made clear the methods for obtaining the equations of motion for these intermediate cases.

NONDIMENSIONALIZATION OF EQUATIONS OF MOTION

The equations of motion may be nondimensionalized in several ways. The use of a method which brings in the conventional stability derivatives of the separate airplanes would be desirable, however. One such method will be illustrated by applying it to equations (21).

In order to nondimensionalize the lateral equations, equation (21.1) is divided by qS, equations (21.2) and (21.3) are divided by qSb, and equation (21.4) is divided by qS_fb_f. Then, the following substitutions are made: $A_1 \equiv \frac{S_fb_f}{Sb}$, $A_2 \equiv \frac{S_fb_f^3}{Sb^3}$, $A_3 \equiv \frac{S_f}{S}$, $A_4 \equiv \frac{b_f}{b}$, and, for the nondimensional time-derivative operator, $D_b \equiv \frac{b}{V_o}\frac{d}{dt}$. The resulting equations are

$$\begin{aligned}&\left(2\mu_b + 4A_1\mu_{b_f}\right)D_b\beta - \left(C_{Y_\beta} + 2A_3C_{Y_{\beta_f}}\right)\beta + \left(2\mu_b + 4A_1\mu_{b_f}\right)D_b\psi - \\ &\qquad \psi\left(C_W + 2A_3C_{W_f}\right)\tan\gamma_o - C_W\phi - 2A_3C_{W_f}\phi_a = C_Y + A_3\left(C_{Y_1} + C_{Y_2}\right)\end{aligned} \tag{22.1}$$

$$\begin{aligned}&-\left(C_{n_\beta} + 2A_1C_{n_{\beta_f}}\right)\beta + \left[2\mu_bK_Z^2 + 4A_2\mu_{b_f}K_{Z_f}^2 + A_1\left(1 + A_4\right)^2\mu_{b_f}\right]D_b^2\psi - \\ &\left[\frac{1}{2} C_{n_r} + A_1A_4C_{n_{r_f}} + \frac{1}{2} A_3\left(1 + A_4\right)^2C_{X_{u'_f}}\right]D_b\psi + 2\mu_bK_{XZ}D_b^2\phi + \\ &\left[\frac{1}{2} A_3\left(1 + A_4\right)C_{X_{\alpha_f}} - \frac{1}{2} C_{n_p}\right]D_b\phi + 4A_2\mu_{b_f}K_{XZ_f}D_b^2\phi_a + \\ &\left[\frac{1}{2} A_3\left(1 + A_4\right)C_{X_{\alpha_f}} - A_1A_4C_{n_{p_f}}\right]D_b\phi_a = C_n + A_1\left(C_{n_1} + C_{n_2}\right) + \\ &\frac{1}{2} A_3\left(1 + A_4\right)\left(C_{X_1} - C_{X_2}\right)\end{aligned} \tag{22.2}$$

$$\begin{aligned}&-C_{l_\beta}\beta + 2\mu_bK_{XZ}D_b^2\psi + \frac{1}{2}\left[A_3\left(1 + A_4\right)C_{Z_{u'_f}} - C_{l_r}\right]D_b\psi + \\ &\left(2\mu_bK_X^2 + A_1\mu_{b_f}\right)D_b^2\phi - \frac{1}{2}\left(C_{l_p} + A_3C_{Z_{\alpha_f}}\right)D_b\phi + A_1A_4\mu_{b_f}D_b^2\phi_a - \\ &\frac{1}{2} A_1C_{Z_{\alpha_f}}D_b\phi_a = C_l - \frac{1}{2} A_3\left(C_{Z_1} - C_{Z_2}\right)\end{aligned} \tag{22.3}$$

$$\begin{aligned}&-C_{l_{\beta_f}}\beta + 2A_4^2\mu_{b_f}K_{XZ_f}D_b^2\psi + \left[\frac{1}{4}\left(1 + A_4\right)C_{Z_{u'_f}} - \frac{1}{2} A_4C_{l_{r_f}}\right]D_b\psi + \frac{1}{2} A_4\mu_{b_f}D_b^2\phi - \\ &\frac{1}{4} C_{Z_{\alpha_f}}D_b\phi + A_4^2\mu_{b_f}\left(2K_{X_f}^2 + \frac{1}{2}\right)D_b^2\phi_a - A_4\left(\frac{1}{2} C_{l_{p_f}} + \frac{1}{4} C_{Z_{\alpha_f}}\right)D_b\phi_a = \\ &\frac{1}{2}\left(C_{l_1} + C_{l_2}\right) - \frac{1}{4}\left(C_{Z_1} - C_{Z_2}\right)\end{aligned} \tag{22.4}$$

In order to nondimensionalize the longitudinal equations, equation (21.5) is divided by qSc, equations (21.6) and (21.7) are divided by qS, and equation (21.8) is divided by qS_fc_f. Then, the following substitutions are made: $B_1 = \frac{S_fc_f}{Sc}$, $B_2 = \frac{S_fc_f^3}{Sc^3}$, $B_3 = \frac{S_f}{S}$, $B_4 = \frac{c_f}{c}$, $B_5 = \frac{b_f}{c}$, $B_6 = \frac{b_f}{c_f}$, and, for the nondimensional time-derivative operator, $D_c = \frac{c}{V_o}\frac{d}{dt}$. The resulting equations are

$$\left(2\mu_cK_Y^2 + 4B_2\mu_{c_f}K_{Y_f}^2\right)D_c^2\theta - \left(\frac{1}{2}C_{m_q} + B_1B_4C_{m_{q_f}}\right)D_c\theta -$$

$$\left(C_{m_{u'}} + 2B_1C_{m_{u'_f}}\right)u' - \left(\frac{1}{2}C_{m_{\dot{\alpha}}} + B_1B_4C_{m_{\dot{\alpha}_f}}\right)D_c\alpha -$$

$$\left(C_{m_\alpha} + 2B_1C_{m_{\alpha_f}}\right)\alpha + \frac{1}{2}B_1B_4B_5C_{m_{\dot{\alpha}_f}}D_c^2\phi_s +$$

$$B_1B_5C_{m_{\alpha_f}}D_c\phi_s = C_m + B_1\left(C_{m_1} + C_{m_2}\right) \tag{22.5}$$

$$\left(C_W + 2B_3C_{W_f}\right)\theta + 2\left(\mu_c + 2B_1\mu_{c_f}\right)D_cu' - \left(C_{X_{u'}} + 2B_3C_{X_{u'_f}}\right)u' -$$

$$\left(C_{X_\alpha} + 2B_3C_{X_{\alpha_f}}\right)\alpha + B_3B_5C_{X_{\alpha_f}}D_c\phi_s = C_X + B_3\left(C_{X_1} + C_{X_2}\right) \tag{22.6}$$

$$-\left(\frac{1}{2}C_{Z_q} + B_1C_{Z_{q_f}} + 2\mu_c + 4B_1\mu_{c_f}\right)D_c\theta + \theta\left(C_W + 2B_3C_{W_f}\right)\tan\gamma_o -$$

$$\left(C_{Z_{u'}} + 2B_3C_{Z_{u'_f}}\right)u' + \left(2\mu_c + 4B_1\mu_{c_f}\right)D_c\alpha - \left(C_{Z_\alpha} + 2B_3C_{Z_{\alpha_f}}\right)\alpha -$$

$$2B_1B_5\mu_{c_f}D_c^2\phi_s + B_3B_5C_{Z_{\alpha_f}}D_c\phi_s = C_Z + B_3\left(C_{Z_1} + C_{Z_2}\right) \tag{22.7}$$

$$B_5\left(\mu_{c_f} + \frac{1}{4} C_{Z_{q_f}}\right)D_c\theta - \frac{1}{2}\theta B_6 C_{W_f} \tan\gamma_o + \frac{1}{2} B_6 C_{Z_{u'_f}} u' - B_5 \mu_{c_f} D_c \alpha +$$

$$\frac{1}{2} B_6 C_{Z_{\alpha_f}} \alpha + B_5{}^2 \mu_{c_f}\left(2K_{X_f}{}^2 + \frac{1}{2}\right)D_c{}^2\phi_s - B_5 B_6\left(\frac{1}{2} C_{l_{p_f}} +\right.$$

$$\left.\frac{1}{4} C_{Z_{\alpha_f}}\right)D_c\phi_s = \frac{1}{2} B_6\left(C_{l_1} - C_{l_2}\right) - \frac{1}{4} B_6\left(C_{Z_1} + C_{Z_2}\right) \qquad (22.8)$$

CONSIDERATION OF AERODYNAMIC INTERFERENCE EFFECTS

Because of the aerodynamic interference between adjacent wings, it is clear that the pressure distributions arising from given motions will be modified. Probably the most important interference effects are the changes in the lift distribution resulting from an angle of attack or rolling velocity. From references 1 and 2 it can be seen that, in addition to modifications of the ordinary stability derivatives, certain unusual stability derivatives are introduced. These derivatives arise because the lift distribution resulting from the motion of a given wing does not vanish at the wing tip, but "spills" over onto the adjacent wing. This phenomenon gives rise to two important dynamic effects. First, there are forces on a given airplane arising from motions of the adjacent one, and, second, there are coupling forces between the lateral and longitudinal modes of the fighters. The second effect arises from the asymmetry of the lift on the fighters. For example, a change in angle of attack on the fighter wing causes a larger change in the lift on the inboard wing than on the outboard wing, because of the interference effect of the adjacent bomber wing. Therefore a rolling moment is produced.

The purpose of this section is not to evaluate these interference effects, but simply to show how they may be brought into the equations of motion when they have been determined by some theoretical or experimental method. In general, it can be seen that the three types of interference which have been mentioned in the discussion of angle-of-attack and rolling-velocity effects (namely, modifications of the ordinary stability derivatives, cross derivatives between the bomber and fighters, and cross derivatives between the fighter lateral and longitudinal modes) are, dynamically, the only types which can occur even in the most general disturbance motions. Therefore, the rolling-velocity and angle-of-attack interference effects will be considered in detail, and, if any other interference effects should turn out to be important, they may be treated in a similar manner.

For this discussion the modifications of the values of the ordinary stability derivatives are unimportant, since they may be taken care of by simply replacing the ordinary derivatives by their modified values. The types of unusual stability derivatives will be distinguished by use of parentheses. For example, the bomber rolling moment due to unit change of fighter angle of attack will be written $(L)_{\alpha_f}$, the fighter rolling moment due to unit bomber angle of attack will be written $\left(L_f\right)_\alpha$, and the fighter rolling moment due to unit angle of attack of the same fighter will be written $\left(L_\alpha\right)_f$.

The unusual stability derivatives arising from fighter angle of attack are $\left(L_\alpha\right)_f$, $(Z)_{\alpha_f}$, and $(L)_{\alpha_f}$. Because of the symmetry, the following definitions may be made:

$$-\left(L_\alpha\right)_1 = \left(L_\alpha\right)_2 \equiv \left(L_\alpha\right)_f \tag{23}$$

$$(Z)_{\alpha_1} = (Z)_{\alpha_2} \equiv (Z)_{\alpha_f} \tag{24}$$

$$(L)_{\alpha_1} = -(L)_{\alpha_2} \equiv (L)_{\alpha_f} \tag{25}$$

Similarly, for the stability derivatives associated with a bomber angle of attack, the following definitions may be used:

$$\left(Z_1\right)_\alpha = \left(Z_2\right)_\alpha \equiv \left(Z_f\right)_\alpha \tag{26}$$

$$-\left(L_1\right)_\alpha = \left(L_2\right)_\alpha \equiv \left(L_f\right)_\alpha \tag{27}$$

For fighter rolling-velocity derivatives

$$-\left(Z_{\dot\phi}\right)_1 = \left(Z_{\dot\phi}\right)_2 \equiv \left(Z_{\dot\phi}\right)_f \tag{28}$$

$$-(Z)_{\dot\phi_1} = (Z)_{\dot\phi_2} \equiv (Z)_{\dot\phi_f} \tag{29}$$

$$(L)_{\dot\phi_1} = (L)_{\dot\phi_2} \equiv (L)_{\dot\phi_f} \tag{30}$$

Finally, for bomber rolling-velocity derivatives,

$$(Z_1)_{\dot{\phi}} = -(Z_2)_{\dot{\phi}} \equiv (Z_f)_{\dot{\phi}} \tag{31}$$

$$(L_1)_{\dot{\phi}} = (L_2)_{\dot{\phi}} \equiv (L_f)_{\dot{\phi}} \tag{32}$$

The forces associated with these derivatives may now be treated exactly as the ordinary aerodynamic forces were. These are simply additional applied forces and may be considered in equation (3), in the proper equations of equations (5), or in the final combined equations of motion. The terms mentioned in equations (23) to (32) enter into equations (5.3), (5.5), (5.9), (5.11), (5.15), and (5.17). Now suppose that it is desired to find the additional terms in the final equations of motion arising from these unconventional stability derivatives. For simplicity, assume the system is of the type described by equations (21). In this case the only final equations which are affected are equations (21.3), (21.4), (21.7), and (21.8).

In equation (21.3), the aerodynamic interaction forces which must be considered may be added to the right-hand side of the equation as additional applied forces. These additional terms are obtained with the use of equations (25), (26), (28), (30), and (31) and are as follows:

$$(L)_{\alpha_f}(\alpha_1 - \alpha_2) + (L)_{\dot{\phi}_f}(\dot{\phi}_1 + \dot{\phi}_2) - \frac{b}{2}\Big[(Z_f)_{\alpha}\alpha - (Z_f)_{\alpha}\alpha -$$

$$(Z_{\dot{\phi}})_f(\dot{\phi}_1 + \dot{\phi}_2) + (Z_f)_{\dot{\phi}}\dot{\phi} + (Z_f)_{\dot{\phi}}\dot{\phi}\Big]$$

These terms have been written out in detail to make clear the method of obtaining the additional terms. By the use of the equations of condition for this case, α_1 and α_2 may be eliminated as follows:

$$\frac{1}{2}\alpha_1 = \frac{w_1}{2V_o} = \frac{1}{2V_o}\left(w - \frac{b}{2}\dot{\phi} - \frac{b_f}{2}\dot{\phi}_1\right) = \frac{\alpha}{2} - \frac{b}{4V_o}\dot{\phi} - \frac{b_f}{2V_o}\frac{\dot{\phi}_1}{2}$$

$$\frac{1}{2}\alpha_2 = \frac{w_2}{2V_o} = \frac{\alpha}{2} + \frac{b}{4V_o}\dot{\phi} + \frac{b_f}{2V_o}\frac{\dot{\phi}_2}{2}$$

[redacted]

and

$$\frac{1}{2}(\alpha_1 + \alpha_2) = \alpha - \frac{b_f}{2V_o}\dot{\phi}_s \tag{33}$$

$$\frac{1}{2}(\alpha_1 - \alpha_2) = -\frac{b}{2V_o}\dot{\phi} - \frac{b_f}{2V_o}\dot{\phi}_a \tag{34}$$

The additional terms to be added to the left-hand side of equation (21.3) are obtained by changing the signs of the previously derived additional applied forces and using equation (34):

$$\frac{1}{V_o}(L)_{\alpha_f}\left(b\dot{\phi} + b_f\dot{\phi}_a\right) - 2(L)_{\dot{\phi}_f}\dot{\phi}_a - b\left(Z_{\dot{\phi}}\right)_f\dot{\phi}_a + b\left(Z_f\right)_{\dot{\phi}}\dot{\phi}$$

A similar process yields the correction terms for the other three modified equations. When the proper collection of terms is made, the additional terms from the rolling-velocity and angle-of-attack aerodynamic interference forces on the system described by equations (21) are

$$\left[\frac{1}{V_o}(L)_{\alpha_f} + (Z_f)_{\dot{\phi}}\right]b\dot{\phi} + \left[\frac{b_f}{V_o}(L)_{\alpha_f} - b\left(Z_{\dot{\phi}}\right)_f - 2(L)_{\dot{\phi}_f}\right]\dot{\phi}_\alpha \tag{21.3'}$$

$$\left[\frac{b_f}{2}(Z_f)_{\dot{\phi}} - \frac{b}{2V_o}(L_\alpha)_f - (L_f)_{\dot{\phi}}\right]\dot{\phi} - \frac{b_f}{2}\left[\left(Z_{\dot{\phi}}\right)_f + \frac{1}{V_o}(L_\alpha)_f\right]\dot{\phi}_a \tag{21.4'}$$

$$-2\left[(Z)_{\alpha_f} + (Z_f)_\alpha\right]\alpha + \left[\frac{b_f}{V_o}(Z)_{\alpha_f} + 2\left(Z_{\dot{\phi}}\right)_f + 2(Z)_{\dot{\phi}_f}\right]\dot{\phi}_s \tag{21.7'}$$

$$\left[(L_\alpha)_f + (L_f)_\alpha + \frac{b_f}{2}(Z_f)_\alpha\right]\alpha - \frac{b_f}{2}\left[\frac{1}{V_o}(L_\alpha)_f + \left(Z_{\dot{\phi}}\right)_f\right]\dot{\phi}_s \tag{21.8'}$$

In practice, if the aerodynamic interference forces had been known, they would have been included in equation (3) as aerodynamic forces in the E_i associated with each degree of freedom. Therefore, they would have appeared in the final equations in the proper form. This more logical procedure was not followed for two reasons: First, no attempt has been made to determine all the interference forces which might be important, since this paper is primarily concerned with the dynamic

[redacted]

rather then the aerodynamic effects of wing-tip coupling. Second, the consideration of aerodynamic interference in the midst of the explanation of the dynamic effects would tend to confuse this explanation.

The additional terms in equations (21.3'), (21.4'), (21.7'), and (21.8') show that the aerodynamic interference terms do not destroy the symmetry of the final equations of the total system, since it will be noticed that the additions to the lateral equations (21.3) and (21.4) contain only lateral degrees of freedom and the additions to the longitudinal equations (21.7) and (21.8) contain only longitudinal variables. The physical reason for this is as follows: Although the aerodynamic interference causes unsymmetrical pressure distributions on the fighters, yet with respect to the plane of symmetry of the total system (that is, the bomber plane of symmetry) these pressure distributions are symmetrical for symmetrical motions and antisymmetrical for antisymmetrical motions.

CONCLUDING REMARKS

A method has been presented for deriving the equations of motion for the small disturbance motions of a symmetrical configuration of wing-tip-coupled airplanes. Lagrange's method of undetermined multipliers was used to take account of the dynamic effects of the constraints at the wing tips on the equations of motion for small disturbances of the unconstrained system of airplanes.

By this method, the equations of motion of a wing-tip-coupled system with three degrees of rotational freedom at each coupling joint were derived with the assumption of no steady-state trim angles between the fighters and the bomber and with the effects of aerodynamic interference between the airplanes being neglected. It was shown that the twelve equations in twelve unknowns, which are necessary to describe this general system, may be separated into mutually independent lateral and longitudinal modes consisting of six equations each. The lateral equations are eleventh order in the time-derivative operator and the longitudinal equations are tenth order. The equations show the importance of the position of the coupling joint in determining the stability derivatives of the combined system. The simple modifications caused by steady-state fighter deflections were also pointed out.

The effects of restricting the rotational motion at the coupling joints were then considered, and it was shown how the equations of motion could be derived for these cases with fewer degrees of freedom. The importance of the effects of the orientation of the rotation axes on the equations of condition in these more restricted cases was pointed out. The equations of motion were then derived for the simplest case of hinge coupling with the hinge axis parallel to the steady-state X-axis and the position of the hinges lying on the common steady-state Y-axis of the airplanes. For this simplest case, the solution of either the lateral or longitudinal equations was shown to be a sixth-order problem. A convenient method for nondimensionalizing the equations of motion was illustrated by applying it to these equations.

Finally, it was shown that the aerodynamic interference forces could be treated in the same manner as the ordinary aerodynamic forces and would introduce no fundamental difficulties.

Since the type of coupling with complete rotational freedom would be most desirable from the point of view of decreasing the strains on the wing structures, it would seem advisable to investigate the stability of this case by use of analog computers. In order to obtain accurate results the aerodynamic interference forces would first have to be evaluated by some theoretical or experimental means. Since the equations of motion are known, the computing machines could then be used to evaluate the effects of varying significant parameters, such as the positions of the coupling joints, on the stability of the system.

Langley Aeronautical Laboratory
National Advisory Committee for Aeronautics
Langley Field, Va.

APPENDIX A

GENERAL EQUATIONS OF MOTION OF A SYMMETRICAL WING-TIP-COUPLED CONFIGURATION OF AIRPLANES WITH COMPLETE ROTATIONAL FREEDOM AT THE COUPLING JOINTS

The lateral equations of motion are obtained as follows by making the conventional transformation to nondimensional translational velocities; that is, $u' \equiv \frac{u}{V_o}$, $\beta \equiv \frac{v}{V_o}$, and $\alpha \equiv \frac{w}{V_o}$, where α and β are, to first order, the disturbance angle of attack and angle of sideslip, respectively:

$$(m + 2m_f)V_o\dot{\beta} - \left(Y_\beta + 2Y_{\beta_f}\right)\beta + 2m_f X\ddot{\psi} + \left(mV_o + 2m_fV_o - \frac{2X}{V_o}Y_{\beta_f}\right)\dot{\psi} -$$

$$\left(W\sin\gamma_o + 2Y_{\beta_f}\right)\psi - 2m_f Z\ddot{\phi} + \frac{2Z}{V_o}Y_{\beta_f}\dot{\phi} - (W\cos\gamma_o)\phi + 2m_fZ_f\ddot{\phi}_a -$$

$$\frac{2Z_f}{V_o}Y_{\beta_f}\dot{\phi}_a - 2(W_f\cos\gamma_o)\phi_a - 2m_fX_f\ddot{\psi}_a + \frac{2X_f}{V_o}Y_{\beta_f}\dot{\psi}_a +$$

$$2\left(Y_{\beta_f} - W_f\sin\gamma_o\right)\psi_a = F_Y + F_{Y_1} + F_{Y_2} \tag{A1}$$

$$-mXV_o\dot{\beta} + \left(XY_\beta - N_\beta\right)\beta + \left(I_Z + m_f\frac{b^2}{2}\right)\ddot{\psi} - \left(N_{\dot{\psi}} + mXV_o + \frac{b^2}{2V_o}X_{u'_f}\right)\dot{\psi} +$$

$$\left(XW\sin\gamma_o\right)\psi - I_{XZ}\ddot{\phi} + \left(\frac{b^2}{2V_o}X_{\alpha_f} - N_{\dot{\phi}}\right)\dot{\phi} + \left(XW\cos\gamma_o\right)\phi + \frac{bb_f}{2V_o}X_{\alpha_f}\dot{\phi}_a +$$

$$m_f\frac{bb_f}{2}\ddot{\psi}_a - \frac{bb_f}{2V_o}X_{u'_f}\dot{\psi}_a - m_fbZ_f\ddot{\theta}_a + \frac{b}{V_o}\left(Z_fX_{u'_f} - X_fX_{\alpha_f}\right)\dot{\theta}_a +$$

$$\left(bW_f\cos\gamma_o - bX_{\alpha_f}\right)\theta_a = N - XF_Y + \frac{b}{2}\left(F_{X_1} - F_{X_2}\right) \tag{A2}$$

$$\begin{aligned}
& mZV_o\dot{\beta} - \left(L_\beta + ZY_\beta\right)\beta - I_{XZ}\ddot{\psi} + \left(mZV_o - L_{\dot{\psi}} + \frac{b^2}{2V_o} Z_{u'_f}\right)\dot{\psi} - \left(ZW \sin \gamma_o\right)\psi + \\
& \quad \left(I_X + m_f \frac{b^2}{2}\right)\ddot{\phi} - \left(L_{\dot{\phi}} + \frac{b^2}{2V_o} Z_{\alpha_f}\right)\dot{\phi} - \left(ZW \cos \gamma_o\right)\phi + m_f \frac{bb_f}{2} \ddot{\phi}_a - \\
& \quad \frac{bb_f}{2V_o} Z_{\alpha_f}\dot{\phi}_a + \frac{bb_f}{2V_o} Z_{u'_f}\dot{\psi}_a - m_f bX_f\ddot{\theta}_a + \left[\frac{b}{V_o}\left(X_f Z_{\alpha_f} - Z_f Z_{u'_f}\right) + \right. \\
& \quad \left. bZ_{\dot{\theta}_f}\right]\dot{\theta}_a + b\left(Z_{\alpha_f} - W_f \sin \gamma_o\right)\theta_a = L + ZF_Y + \frac{b}{2}\left(F_{Z_2} - F_{Z_1}\right) \qquad \text{(A3)}
\end{aligned}$$

$$\begin{aligned}
& m_f Z_f V_o\dot{\beta} - \left(L_{\beta_f} + Z_f Y_{\beta_f}\right)\beta + m_f X Z_f\ddot{\psi} + \left[\frac{1}{V_o}\left(\frac{bb_f}{4} Z_{u'_f} - XL_{\beta_f} - XZ_f Y_{\beta_f}\right) + \right. \\
& \quad \left. m_f Z_f V_o\right]\dot{\psi} - \left(Z_f Y_{\beta_f} + L_{\beta_f}\right)\psi + m_f\left(\frac{bb_f}{4} - ZZ_f\right)\ddot{\phi} + \frac{1}{V_o}\left(ZZ_f Y_{\beta_f} + \right. \\
& \quad \left. ZL_{\beta_f} - \frac{bb_f}{4} Z_{\alpha_f}\right)\dot{\phi} + \left(m_f Z_f^{\,2} + m_f \frac{b_f^{\,2}}{4} + I_{X_f}\right)\ddot{\phi}_a - \left[\frac{1}{V_o}\left(Z_f^{\,2} Y_{\beta_f} + \right.\right. \\
& \quad \left.\left. Z_f L_{\beta_f} + \frac{b_f^{\,2}}{4} Z_{\alpha_f}\right) + L_{\dot{\phi}_f}\right]\dot{\phi}_a - \left(Z_f W_f \cos \gamma_o\right)\phi_a - \left(I_{XZ_f} + \right. \\
& \quad \left. m_f X_f Z_f\right)\ddot{\psi}_a + \left[\frac{1}{V_o}\left(X_f L_{\beta_f} + X_f Z_f Y_{\beta_f} + \frac{b_f^{\,2}}{4} Z_{u'_f}\right) - L_{\dot{\psi}_f}\right]\dot{\psi}_a + \\
& \quad \left(L_{\beta_f} + Z_f Y_{\beta_f} - Z_f W_f \sin \gamma_o\right)\psi_a - m_f \frac{b_f}{2} X_f\ddot{\theta}_a + \left[\frac{b_f}{2V_o}\left(X_f Z_{\alpha_f} - \right.\right. \\
& \quad \left.\left. Z_f Z_{u'_f}\right) + \frac{b_f}{2} Z_{\dot{\theta}_f}\right]\dot{\theta}_a + \frac{b_f}{2}\left(Z_{\alpha_f} - W_f \sin \gamma_o\right)\theta_a = \\
& \quad \frac{1}{2}\left[\left(L_1 + L_2\right) + Z_f\left(F_{Y_1} + F_{Y_2}\right) - \frac{b_f}{2}\left(F_{Z_1} - F_{Z_2}\right)\right] \qquad \text{(A4)}
\end{aligned}$$

$$
\begin{aligned}
&-m_f X_f V_o \dot{\beta} + \left(X_f Y_{\beta_f} - N_{\beta_f}\right)\beta + m_f\left(\frac{bb_f}{4} - XX_f\right)\ddot{\psi} + \left[\frac{1}{V_o}\left(XX_f Y_{\beta_f} - XN_{\beta_f} - \right.\right.\\
&\left.\left.\frac{bb_f}{4} X_{u'_f}\right) - m_f X_f V_o\right]\dot{\psi} + \left(X_f Y_{\beta_f} - N_{\beta_f}\right)\psi + m_f X_f Z\ddot{\phi} + \frac{1}{V_o}\left(ZN_{\beta_f} - ZX_f Y_{\beta_f} + \right.\\
&\left.\frac{bb_f}{4} X_{\alpha_f}\right)\dot{\phi} - \left(I_{XZ_f} + m_f X_f Z_f\right)\ddot{\phi}_a + \left[\frac{1}{V_o}\left(\frac{b_f^2}{4} X_{\alpha_f} + Z_f X_f Y_{\beta_f} - Z_f N_{\beta_f}\right) - \right.\\
&\left.N_{\dot{\phi}_f}\right]\dot{\phi}_a + \left(X_f W_f \cos \gamma_o\right)\phi_a + \left(I_{Z_f} + m_f X_f^2 + m_f \frac{b_f^2}{4}\right)\ddot{\psi}_a + \left[\frac{1}{V_o}\left(X_f N_{\beta_f} - \right.\right.\\
&\left.\left.X_f^2 Y_{\beta_f} - \frac{b_f^2}{4} X_{u'_f}\right) - N_{\dot{\psi}_f}\right]\dot{\psi}_a + \left(N_{\beta_f} - X_f Y_{\beta_f} + X_f W_f \sin \gamma_o\right)\psi_a - \\
&m_f Z_f \frac{b_f}{2} \ddot{\theta}_a + \frac{b_f}{2V_o}\left(Z_f X_{u'_f} - X_f X_{\alpha_f}\right)\dot{\theta}_a + \frac{b_f}{2}\left(W_f \cos \gamma_o - X_{\alpha_f}\right)\theta_a = \\
&\frac{1}{2}\left[\left(N_1 + N_2\right) + \frac{b_f}{2}\left(F_{X_1} - F_{X_2}\right) - X_f\left(F_{Y_1} + F_{Y_2}\right)\right]
\end{aligned}
\tag{A5}
$$

$$
\begin{aligned}
&-m_f Z_f \frac{b}{2} \ddot{\psi} + \frac{b}{2V_o}\left(Z_f X_{u'_f} - X_f Z_{u'_f} - M_{u'_f}\right)\dot{\psi} + \frac{b}{2}\left(\frac{1}{V_o} M_{\dot{\alpha}_f} - m_f X_f\right)\ddot{\phi} + \\
&\frac{b}{2V_o}\left(M_{\alpha_f} + X_f Z_{\alpha_f} - Z_f X_{\alpha_f}\right)\dot{\phi} + \frac{b_f}{2}\left(\frac{1}{V_o} M_{\dot{\alpha}_f} - m_f X_f\right)\ddot{\phi}_a + \frac{b_f}{2V_o}\left(M_{\alpha_f} + X_f Z_{\alpha_f} - \right.\\
&\left.Z_f X_{\alpha_f}\right)\dot{\phi}_a - m_f Z_f \frac{b_f}{2} \ddot{\psi}_a + \frac{b_f}{2V_o}\left(Z_f X_{u'_f} - X_f Z_{u'_f} - M_{u'_f}\right)\dot{\psi}_a + \left(I_{Y_f} + \right.\\
&\left.m_f X_f^2 + m_f Z_f^2 - \frac{X_f}{V_o} M_{\dot{\alpha}_f}\right)\ddot{\theta}_a + \left[\frac{1}{V_o}\left(X_f Z_f X_{\alpha_f} + X_f Z_f Z_{u'_f} - X_f^2 Z_{\alpha_f} - \right.\right.\\
&\left.\left.Z_f^2 X_{u'_f} - X_f M_{\alpha_f} + Z_f M_{u'_f}\right) - X_f Z_{\dot{\theta}_f} - M_{\dot{\alpha}_f} - M_{\dot{\theta}_f}\right]\dot{\theta}_a + \left(Z_f X_{\alpha_f} - X_f Z_{\alpha_f} - \right.\\
&\left.M_{\alpha_f} - Z_f W_f \cos \gamma_o + X_f W_f \sin \gamma_o\right)\theta_a = \frac{1}{2}\left[\left(M_1 - M_2\right) - Z_f\left(F_{X_1} - F_{X_2}\right) + \right.\\
&\left.X_f\left(F_{Z_1} - F_{Z_2}\right)\right]
\end{aligned}
\tag{A6}
$$

The longitudinal equations for this general case are

$$
\begin{aligned}
&(m + 2m_f)V_o\dot{u}' - \left(X_{u'} + 2X_{u'_f}\right)u' - \left(X_\alpha + 2X_{\alpha_f}\right)\alpha + 2m_f Z\ddot{\theta} + \frac{2}{V_o}\Big(XX_{\alpha_f} - \\
&\quad ZX_{u'_f}\Big)\dot{\theta} + \left(W \cos \gamma_o + 2X_{\alpha_f}\right)\theta + \frac{b_f}{V_o} X_{\alpha_f}\dot{\phi}_s + m_f b_f\ddot{\psi}_s - \frac{b_f}{V_o} X_{u'_f}\dot{\psi}_s - \\
&\quad 2m_f Z_f\ddot{\theta}_s + \frac{2}{V_o}\left(Z_f X_{u'_f} - X_f X_{\alpha_f}\right)\dot{\theta}_s + 2\left(W_f \cos \gamma_o - X_{\alpha_f}\right)\theta_s = \\
&\quad F_X + F_{X_1} + F_{X_2}
\end{aligned}
\tag{A7}
$$

$$
\begin{aligned}
&-\left(Z_{u'} + 2Z_{u'_f}\right)u' + (m + 2m_f)V_o\dot{\alpha} - \left(Z_\alpha + 2Z_{\alpha_f}\right)\alpha - 2m_f X\ddot{\theta} + \Bigg[\frac{2}{V_o}\Big(XZ_{\alpha_f} - \\
&\quad ZZ_{u'_f}\Big) - mV_o - 2m_f V_o - Z_{\dot{\theta}}\Bigg]\dot{\theta} + \left(W \sin \gamma_o + 2Z_{\alpha_f}\right)\theta - m_f b_f\ddot{\phi}_s + \\
&\quad \frac{b_f}{V_o} Z_{\alpha_f}\dot{\phi}_s - \frac{b_f}{V_o} Z_{u'_f}\dot{\psi}_s + 2m_f X_f\ddot{\theta}_s + 2\left[\frac{1}{V_o}\left(Z_f Z_{u'_f} - X_f Z_{\alpha_f}\right) - Z_{\dot{\theta}_f}\right]\dot{\theta}_s + \\
&\quad 2\left(W_f \sin \gamma_o - Z_{\alpha_f}\right)\theta_s = F_Z + F_{Z_1} + F_{Z_2}
\end{aligned}
\tag{A8}
$$

$$
\begin{aligned}
&-mZV_o\dot{u}' + \left(ZX_{u'} - XZ_{u'} - M_{u'}\right)u' + \left(mXV_o - M_{\dot{\alpha}}\right)\dot{\alpha} + \left(ZX_\alpha - XZ_\alpha - M_\alpha\right)\alpha + \\
&\quad I_Y\ddot{\theta} - \left(M_{\dot{\theta}} + XZ_{\dot{\theta}} + mXV_o\right)\dot{\theta} + \left(XW \sin \gamma_o - ZW \cos \gamma_o\right)\theta = \\
&\quad M + XF_Z - ZF_X
\end{aligned}
\tag{A9}
$$

$$
\begin{aligned}
&\frac{b_f}{2} Z_{u'_f} u' - m_f \frac{b_f}{2} V_o \dot{\alpha} + \frac{b_f}{2} Z_{\alpha_f} \alpha + m_f \frac{b_f}{2} X \ddot{\theta} + \left[\frac{b_f}{2V_o}\left(Z Z_{u'_f} - X Z_{\alpha_f}\right) + \right. \\
&\left. m_f \frac{b_f}{2} V_o\right]\dot{\theta} - \frac{b_f}{2} Z_{\alpha_f} \theta + \left(I_{X_f} + m_f Z_f^2 + m_f \frac{b_f^2}{4}\right)\ddot{\phi}_s - \left[\frac{1}{V_o}\left(Z_f L_{\beta_f} + \right.\right. \\
&\left.\left. Z_f^2 Y_{\beta_f} + \frac{b_f^2}{4} Z_{\alpha_f}\right) + L_{\dot{\phi}_f}\right]\dot{\phi}_s - \left(Z_f W_f \cos \gamma_o\right)\phi_s - \left(I_{XZ_f} + m_f X_f Z_f\right)\ddot{\psi}_s + \\
&\left[\frac{1}{V_o}\left(X_f L_{\beta_f} + Z_f X_f Y_{\beta_f} + \frac{b_f^2}{4} Z_{u'_f}\right) - L_{\dot{\psi}_f}\right]\dot{\psi}_s + \left(L_{\beta_f} + Z_f Y_{\beta_f} - \right. \\
&\left. Z_f W_f \sin \gamma_o\right)\psi_s - m_f X_f \frac{b_f}{2} \ddot{\theta}_s + \left[\frac{b_f}{2V_o}\left(X_f Z_{\alpha_f} - Z_f Z_{u'_f}\right) + \frac{b_f}{2} Z_{\dot{\theta}_f}\right]\dot{\theta}_s + \\
&\frac{b_f}{2}\left(Z_{\alpha_f} - W_f \sin \gamma_o\right)\theta_s = \frac{1}{2}\left[\left(L_1 - L_2\right) + Z_f\left(F_{Y_1} - F_{Y_2}\right) - \right. \\
&\left. \frac{b_f}{2}\left(F_{Z_1} + F_{Z_2}\right)\right] \qquad \text{(A10)}
\end{aligned}
$$

$$
\begin{aligned}
&m_f \frac{b_f}{2} V_o \dot{u}' - \frac{b_f}{2} X_{u'_f} u' - \frac{b_f}{2} X_{\alpha_f} \alpha + m_f \frac{b}{2} Z \ddot{\theta} + \frac{b_f}{2V_o}\left(X X_{\alpha_f} - Z X_{u'_f}\right)\dot{\theta} + \\
&\frac{b_f}{2} X_{\alpha_f} \theta - \left(I_{XZ_f} + m_f X_f Z_f\right)\ddot{\phi}_s + \left[\frac{1}{V_o}\left(\frac{b_f^2}{4} X_{\alpha_f} + X_f Z_f Y_{\beta_f} - Z_f N_{\beta_f}\right) - \right. \\
&\left. N_{\dot{\phi}_f}\right]\dot{\phi}_s + \left(X_f W_f \cos \gamma_o\right)\phi_s + \left(I_{Z_f} + m_f X_f^2 + m_f \frac{b_f^2}{4}\right)\ddot{\psi}_s + \left[\frac{1}{V_o}\left(X_f N_{\beta_f} - \right.\right. \\
&\left.\left. X_f^2 Y_{\beta_f} - \frac{b_f^2}{4} X_{u'_f}\right) - N_{\dot{\psi}_f}\right]\dot{\psi}_s + \left(N_{\beta_f} - X_f Y_{\beta_f} + X_f W_f \sin \gamma_o\right)\psi_s - \\
&m_f \frac{b_f}{2} Z_f \ddot{\theta}_s + \frac{b_f}{2V_o}\left(Z_f X_{u'_f} - X_f X_{\alpha_f}\right)\dot{\theta}_s + \frac{b_f}{2}\left(W_f \cos \gamma_o - X_{\alpha_f}\right)\theta_s = \\
&\frac{1}{2}\left[\left(N_1 - N_2\right) + \frac{b_f}{2}\left(F_{X_1} + F_{X_2}\right) - X_Y\left(F_{Y_1} - F_{Y_2}\right)\right] \qquad \text{(A11)}
\end{aligned}
$$

$$
\begin{aligned}
&-m_f Z_f V_o \dot{u}' + (Z_f X_{u'_f} - X_f Z_{u'_f} - M_{u'_f})u' + (m_f X_f V_o - M_{\dot{\alpha}_f})\dot{\alpha} + (Z_f X_{\alpha_f} - \\
&X_f Z_{\alpha_f} - M_{\alpha_f})\alpha + \left(\frac{X}{V_o} M_{\dot{\alpha}_f} - m_f Z Z_f - m_f X X_f\right)\ddot{\theta} + \Bigl[\frac{1}{V_o}(Z Z_f X_{u'_f} - Z_f X X_{\alpha_f} - \\
&X_f Z Z_{u'_f} + X_f X Z_{\alpha_f} - Z M_{u'_f} + X M_{\alpha_f}) - m_f X_f V_o + M_{\dot{\alpha}_f}\Bigr]\dot{\theta} + (X_f Z_{\alpha_f} - \\
&Z_f X_{\alpha_f} + M_{\alpha_f})\theta + \frac{b_f}{2}\left(\frac{1}{V_o} M_{\dot{\alpha}_f} - m_f X_f\right)\ddot{\phi}_s + \frac{b_f}{2V_o}(X_f Z_{\alpha_f} - Z_f X_{\alpha_f} + M_{\alpha_f})\dot{\phi}_s - \\
&m_f Z_f \frac{b_f}{2}\ddot{\psi}_s + \frac{b_f}{2V_o}(Z_f X_{u'_f} - X_f Z_{u'_f} - M_{u'_f})\dot{\psi}_s + (I_{Y_f} + m_f Z_f^2 + m_f X_f^2 - \\
&\frac{X_f}{V_o} M_{\dot{\alpha}_f})\ddot{\theta}_s + \Bigl[-M_{\dot{\theta}_f} - M_{\dot{\alpha}_f} - X_f Z_{\dot{\theta}_f} + \frac{1}{V_o}(X_f Z_f X_{\alpha_f} + X_f Z_f Z_{u'_f} - X_f^2 Z_{\alpha_f} - \\
&Z_f^2 X_{u'_f} + Z_f M_{u'_f} - X_f M_{\alpha_f})\Bigr]\dot{\theta}_s + (Z_f X_{\alpha_f} - X_f Z_{\alpha_f} - Z_f W_f \cos\gamma_o + \\
&X_f W_f \sin\gamma_o - M_{\alpha_f})\theta_s = \frac{1}{2}\Bigl[(M_1 + M_2) - Z_f(F_{X_1} + F_{X_2}) + X_f(F_{Z_1} + F_{Z_2})\Bigr]
\end{aligned}
\tag{A12}
$$

REFERENCES

1. Anderson, Clarence E., Jr.: Pilot's Comments on Wing-Tip Coupling of Aircraft. MR No. MCRFT-2283, Air Materiel Command, Flight Test Div., U. S. Air Force, April 20, 1950.

2. Leatherbury, C. H., Jr.: Wind-Tunnel Tests of a Tapered Model Wing with a Reflection Plane at One Wing Tip. Rep. C-333 Aero 792, David Taylor Model Basin, Navy Dept., July 1950.

3. Bennett, Charles V., and Cadman, Robert B.: Free-Flight-Tunnel Investigation of the Dynamic Lateral Stability and Control Characteristics of a Tip-to-Tip Bomber-Fighter Coupled Airplane Configuration. NACA RM L51A12, 1951.

4. Diederich, Franklin W., and Zlotnick, Martin: Calculated Lift Distributions of a Consolidated Vultee B-36 and Two Boeing B-47 Airplanes Coupled at the Wing Tips. NACA RM L50I26, 1950.

5. Joos, Georg: Theoretical Physics. G. E. Stechert & Co. (New York), 1934, pp. 106-111.

6. Webster, Arthur Gordon: The Dynamics of Particles and of Rigid, Elastic, and Fluid Bodies. Third ed., G. E. Stechert & Co. (New York), 1942, pp. 57-65 and 313-316.

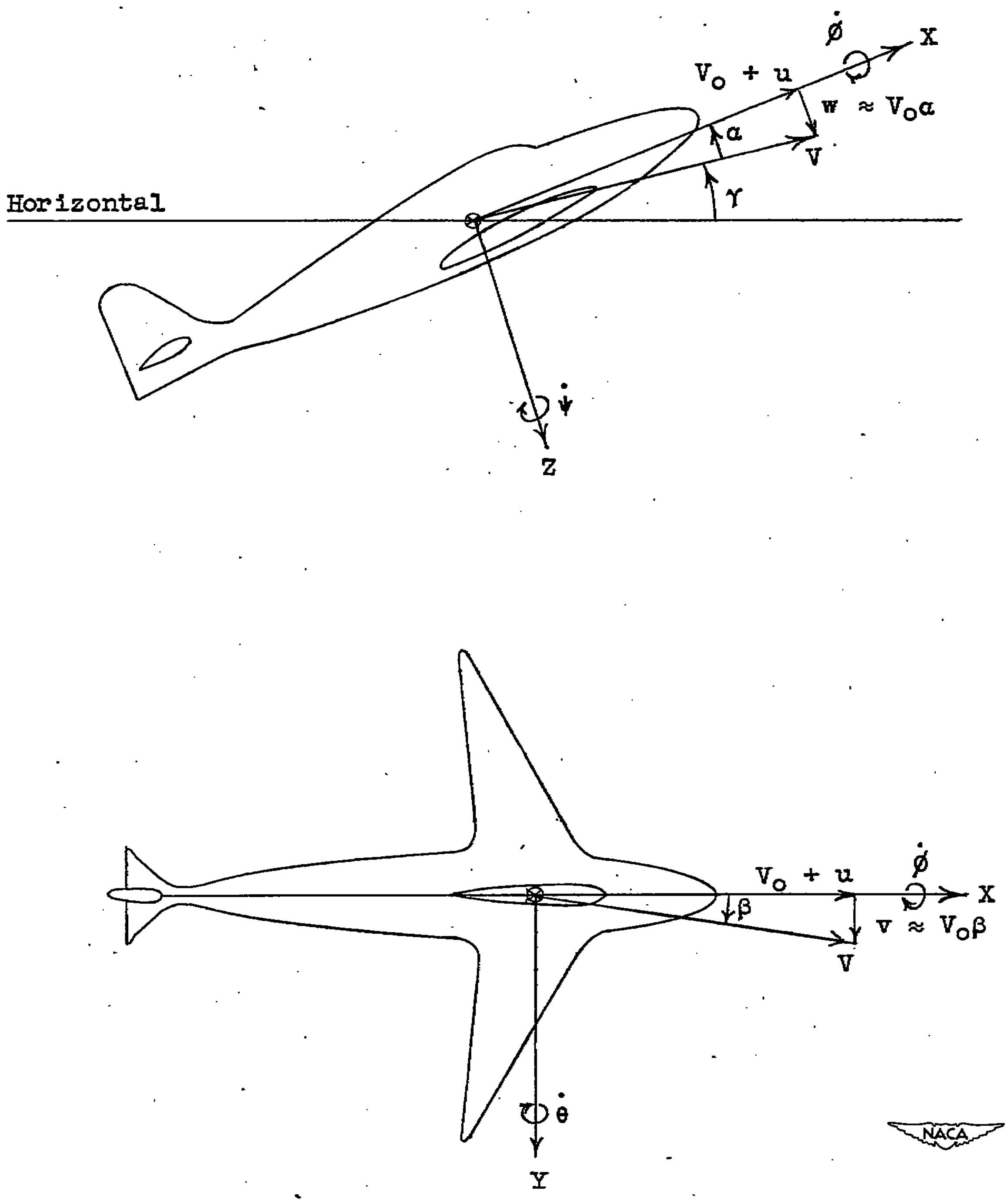

Figure 1.- Stability axes. The motion of each individual airplane is given in terms of stability axes in that airplane. Arrows indicate positive direction.

[redacted]